

# Gonality of Points in Brill-Noether Loci of the Moduli Space of Curves

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# Preface

In this master thesis we want to study the geometry of the Brill-Noether locus  $\mathcal{M}_{d,g}^r$ . A typical problem is to find the gonality of a point  $[C] \in \mathcal{M}_{d,g}^r$ . In general, this is a very hard problem, because this scheme has many components, some of them are reduced and may not be of expected dimension. Therefore we restrict ourself to look at “nice” components of  $\mathcal{M}_{g,d}^r$ , i.e. a component which is generically smooth, of the expected dimension and with point corresponding to a curve with a very ample  $\mathfrak{g}_d^r$ . We will calculate the gonality of “nice” points of the Brill-Noether locus  $\mathcal{M}_{d,g}^r$ , each point representing a smooth curve  $C \subseteq \mathbb{P}^r$  of degree  $d$  and genus  $g$ .

The thesis is divided into two parts; In the first part, we are interested in the gonality of curves in  $\mathbb{P}^r$  which can be realized as  $K3$  sections. In the second part, we will find out what curves on the three complete intersections  $S_4 \subseteq \mathbb{P}^3$ ,  $S_{2,3} \subseteq \mathbb{P}^4$  and  $S_{2,2,2} \subseteq \mathbb{P}^5$  corresponds to smooth points in the Hilbert scheme.

The motivation for this thesis is mainly from the work of G. Farkas [Far01], where he gives explicit formulas for the gonality of curves in  $\mathbb{P}^r$  which can be seen as sections on  $K3$  surfaces under some restrictions and he shows what curves on the quartic  $K3$  surface in  $\mathbb{P}^3$ , under almost the same restrictions, that corresponds smooth points in the Hilbert scheme.

We will generalize Farkas’ result about gonality of  $K3$  sections and we will study  $K3$  surfaces equipped with a Picard group of rank 3, which is not studied before. In the main article, [Far01], Farkas assumes no rational curves. The understanding of why he eliminates these curves, gives me good reason to say that the rational curves are the “bad guys” in this field, as we will see in Chapter 6. Even though I have been struggling to gain control over these curves, I must say that it is fascinating how the simplest curves are causing so much trouble.

This thesis is organized as follows:

Chapter 1 gives a brief introduction to the history of  $K3$  surfaces and we will see how they arise in theoretical physics. In the end we give some basic definitions and notations we will use in this thesis.

Chapter 2 gives an introduction to geometry of surfaces, and should be understandable for anyone with some basic knowledge in algebraic geometry. We will include some basic facts about the tools we will use frequently in later chapters.

In Chapter 3 we limit ourself to the study of geometry on  $K3$  surfaces, which is the main topic of this thesis. We will give some of the basic properties of this family of surfaces and as well as provide some well-known conjectures which have been studied a lot on  $K3$  surfaces. Finally we will see some examples.

Chapter 4 is devoted to some result on  $K3$  surfaces, which we will use later. Corollary 4.9 is a new result about exceptional curves on  $K3$  surfaces.

Chapter 5 is Brill-Noether theory, the study of nonspecial linear systems on curves. We study the minimal degree of curves on rank 2  $K3$  surfaces to the projective line. The main result is Theorem 5.1, where we generalize Farkas' result [Far01, Theorem 3] about gonality of  $K3$  sections. From this theorem follows Example 5.2 and Corollary 5.3.

In Chapter 6 we take one step further from Chapter 5. We add one more generator to the Picard group, so that we have 3 generators instead of 2. The main results are Proposition 6.3, Theorem 6.5 and Corollary 6.6. In Proposition 6.1 we show existence of such a  $K3$  surface, under some restrictions, and in Proposition 6.3 we show existence of a hyperelliptic curve and a tetragonal curve. In Theorem 6.5 we calculate the gonality of curves in  $\mathbb{P}^r$  which has a base point free complete linear system, and in Corollary 6.6 we generalize Proposition 6.3. The reason Proposition 6.3 is a part of the thesis is because its proof is different from the generalized result and we will make use of some parts of the proof in Chapter 7, when we need the parity of  $(d, g)$  to eliminate  $-2$ -curves.

In Chapter 7 we study curves on the three complete intersections  $S_4$ ,  $S_{2,3}$  and  $S_{2,2,2}$ , with Picard rank 2 and 3, and shows when the curves corresponds to smooth points in the Hilbert scheme. The main results are Lemma 7.5, Lemma 7.6 and Corollary 7.8. When we are in the Picard rank 2 case, Lemma 7.5 gives the numerical conditions for what curves on the three complete intersections corresponds to smooth points in the Hilbert scheme. When we are in Picard rank 3 case, Lemma 7.6 gives the numerical conditions for when a line bundle, on the  $K3$  surfaces studied in Chapter 6, is nonspecial.

In Corollary 7.8 we study smooth points in the Hilbert scheme, where each point representing a smooth curve on the two complete intersections,  $S_4$  and  $S_{2,2,2}$ , each with Picard group of rank 3.

Chapter 8 is devoted to unfinished work and some ideas of how to approach some of the results obtained, from a different angle.





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# Chapter 1

## Background

### 1.1 Motivation and Application

Euclidean geometry was developed by the greek mathematician Euclid around 300 B.C., in his famous book “*Elements*”. This way of looking at shapes and the relations between them ruled for many centuries. In the 18th century, mathematicians started to think of a generalization of the Euclidean space. They noticed that it may not happen that the sum of the angles of a triangle is  $180^\circ$  and that parallel lines could in fact cross each other, so the conclusion was that the geometry of the shapes was completely dependent of the curvature of the space they were living in. This idea was the motivational building block of algebraic geometry and differential geometry.

In 1915, the german physicist Albert Einstein took this idea of curved space to a completely new level and made it so important that he revolutionized the way we think of the universe. The theory is known as the *General Theory of Relativity*. He showed that we do not live in a 3-dimensional Euclidean space, but in a 4-dimensional curved space, which he called *Spacetime*. The General Theory of Relativity is a beautiful application of non-Euclidean geometry.

In the same period as Albert Einstein published the General Theory of Relativity, interesting things started to happend on a much smaller scale. Physicists made experiments which showed that strange things happend on atomic level, and they couldn’t explain these results by using classical mechanics, so a new set of rules had to be established. This lead to a new field called *Quantum Mechanics*. After the technology was getting more and

more advanced one could find that the atoms was not the smallest building blocks in the universe, but elementary particles such as quarks and leptons. The interesting thing is that Einstein's explanation of how gravity works cannot make any predictions in the quantum world. Physicists have tried to reconcile quantum mechanics and general relativity for many years without succeeding.

In the middle of the 19th century, a new theory was developed. Theoretical physicists likes to call it *String Theory*. It says that elementary particles are not made of 0-dimensional objects, but 1-dimensional strings oscillating on a Calabi-Yau manifold. String Theory is considered as the best candidate for the *Theory of Everything* (TOE). Physics is an experimental science, so in order to verify a theory, it must be proven experimentally. No one has yet been able to test String Theory against the Standard Model<sup>1</sup>. The problem is that the strings are much smaller than any elementary particle<sup>2</sup> and string theory requires that spacetime have 11 dimensions [DLM95]<sup>3</sup>.

A  $K3$  surface is a 2-dimensional Calabi-Yau manifold. Although it is the 3-dimensional Calabi-Yau manifolds that are in a direct link with string theory, one can use  $K3$  surfaces to study for example string duality<sup>4</sup>, see [Asp96]. In fact, when one studies string duality,  $K3$  surfaces are one of the spaces that always arise. Mathematicians have been studying the geometry of  $K3$  surfaces for a long time. Physicists was not interested in  $K3$  surfaces<sup>5</sup> until Yau proved the Calabi's conjecture in 1977 [Yau77]. Since then  $K3$  surfaces have been a "model toy" for compactifications (see for example [DNP83]) as it provides the second simplest example of a Ricci-flat compact manifold after the torus. Another reason for study  $K3$  surfaces is that the mathematics of heterotic string<sup>6</sup> appears to be intrinsically bound by the geometry of the  $K3$  surface. When the heterotic string appears on one side of pair of dual

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<sup>1</sup>The *Standard Model* is a theory describing the three fundamental forces, i.e. the *weak nuclear force*, the *strong nuclear force* and the *electromagnetic force* and the *subatomic particles*.

<sup>2</sup>A comparison with the size of an electron with approximately diameter  $5,63 \cdot 10^{-15}$ , a string is a loop of diameter  $10^{-35}$ m, which is almost the *Planck length* of  $1,616199 \cdot 10^{-35}$ m.

<sup>3</sup>The number of dimensions comes from  $M$ -theory, which is a theory that unifies the 5 string theories into a superstring theory. For more information, see the work of Edward Witten, the father of  $M$ -theory.

<sup>4</sup>The notion of duality is used in physics, when two theories explains the same physics.

<sup>5</sup>See for example [HP78] for an early paper.

<sup>6</sup>A *heterotic string*, in string theory, is a mixture of the bosonic string and the superstring.

theories, the  $K3$  surface is likely to occur.

## 1.2 History of $K3$ Surfaces

The history of  $K3$  surfaces goes back to classical algebraic geometry. The name “ $K3$  surface” was coined by A. Weil in his famous “Final report on research contract” [Wei80]. In his comments on this report Weil writes:

*Dans la seconde partie de mon rapport, il s'agit des variétés  
kählériennes dites  $K3$ , ainsi nommées en l'honneur de Kummer,  
Kodaira, Kähler et de la belle montagne  $K2$  au Cachmire.*

In the very same report the following conjectures, due to Andreotti and Weil, were stated:

- i)  $K3$  surfaces form one family;
- ii) all  $K3$  surfaces are Kähler;
- iii) the period map is surjective;
- iv) a form of global Torelli theorem holds.

Now all these questions have been answered positively. Conjecture i) was proved by Kodaira [Kod64]. Conjecture ii) was first shown by Siu [Siu83, Section 14]. The surjectivity of the period map was proved by general Kähler  $K3$  surfaces by Todorov [Tod89] and Looijenga [Loo81] (Conjecture iii)). Piatetskii-Šapiro and Šhafarevich proved the Strong Torelli theorem for algebraic  $K3$  surfaces in [Pv71] and in general by Burns and Papoport in [BR75], which proved the conjecture iv) for Kähler  $K3$  surfaces, and hence for all  $K3$ s.

## 1.3 Basic Definitions

A *surface* will mean a compact connected 2-dimensional manifold over  $\mathbb{C}$ . A *curve*  $C$  will be a reduced and irreducible complex analytic space of dimension 1. On a curve (resp. surface), a divisor will be a formal sum of points (resp. curves) counted with multiplicities. A divisor  $D = \sum n_\Gamma \Gamma$  on a variety  $X$  is called *effective* if each  $n_\Gamma \geq 0$ , usually written  $D \geq 0$ . A curve on a surface

will be any effective divisor (one component) on the surface. A point  $P$  will mean a closed point, unless otherwise specified.

By a  $K3$  surface<sup>7</sup> we mean a nonsingular surface  $S$  such that  $H^1(S, \mathcal{O}_S) = 0$  and the canonical divisor  $K_S$  on  $S$  is trivial, i.e.  $K_S \sim 0$ . Throughout this thesis we will write  $S$  to be a  $K3$  surface, and  $X$  will be a variety/scheme.

It should be noted, that Weil's definition of a  $K3$  surface was different from the standard definition used nowadays. He defined it in the following way: A surface is  $K3$  if its underlying differentiable structure was that of a quartic surface in  $\mathbb{P}^3$ . Seiberg-Witten theory shows that any compact complex surface diffeomorphic to a quartic is a  $K3$  surface. The set of  $K3$  surfaces over a field with characteristic 0 is a 20-dimensional family<sup>8</sup>, all diffeomorphic to one another.

*The family of  $K3$  surfaces having  $k$  or more divisors independent in homology forms a dense countable union of subvarieties of dimension  $20 - k$  in the family of all  $K3$ s; in particular, on the generic algebraic  $K3$  surface all divisors are homologous to multiples of the hyperplane class.*

A singular  $K3$  surface is an algebraic  $K3$  surface whose Picard rank equals the maximum possible number 20. The betti numbers of  $K3$ s are 1, 0, 22, 0 and 1. All complex  $K3$  surfaces are diffeomorphic, so they have the same betti numbers. We can define the Hodge numbers of a space  $X$  as the dimension of the Dolbeault cohomology groups<sup>9</sup>

$$h^{p,q}(X) = \dim H^{p,q}(X)$$

Using the properties of a  $K3$  surface  $S$ , we can easily find all  $h^{p,q}(S)$ s for  $0 \leq p, q \leq 2$  (see [Asp96, Section 2.1]). Hence it gives the characteristic Hodge diamond,

$$\begin{array}{ccccc} & & h^{0,0} & & 1 \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & 1 \end{array} = \begin{array}{ccc} & 0 & 0 \\ 1 & 20 & 1 \\ & 0 & 0 \end{array}$$

<sup>7</sup>In this thesis we will be concerned with  $K3$  surfaces in characteristic 0. For  $K3$  surfaces in positive characteristic, see for example the work of M. Artin [Art74].

<sup>8</sup>In positive characteristic, the family of  $K3$  surfaces are 22-dimensional. A  $K3$  surfaces with Picard rank 22 is called *supersingular*.

<sup>9</sup>*Dolbeault cohomology* is the analogue of de Rham cohomology for complex manifolds.

Notice that we can read off the betti number, sum adding the numbers in each row. In higher dimensions,  $K3$  surfaces can be generalized in two ways; namely as Calabi-Yau manifolds or irreducible symplectic manifolds (i.e. hyperkähler manifolds).





# Chapter 2

## General Knowledge

We will consider this chapter and the next as an introduction to this thesis, and should be understandable for anyone who has taken a course in commutative algebra and basic algebraic geometry. The main references from this chapter are from [Har77], [Per00] and [Mil96].

### 2.1 Geometry on a Surface

Let  $X$  be a surface containing the curves  $C, D$ . Then we define the intersection number  $C.D$  to be the following: For a point  $P \in C \cap D$  the *intersection multiplicity*  $(C.D)_P$  of  $C$  and  $D$  at  $P$  is defined to be the length of the  $\mathcal{O}_P$ -module  $\mathcal{O}_P/(f, g)$ , where  $C = V(f)$  and  $D = V(g)$ . Hence,

$$C.D = \sum_{P \in C \cap D} \text{length } \mathcal{O}_P/(f, g).$$

We say that  $C$  and  $D$  meet *transversally at  $P$*  if the ideal generated by the local equations  $g, f = 0$  is maximal in the local ring  $\mathcal{O}_P$ . For example, if  $C, D$  are curves meeting transversally in  $s$  points, then clearly  $C.D = s$ . Linear equivalence is an important concept when we work with divisors.  $C$  is linearly equivalent to  $D$ ,  $C \sim D$ , if  $C - D$  is a *principal divisor*, which is denoted as the zero divisor, see [Har77, p.131].

If  $D_1, D_2 \geq 0$  are effective divisors with no common components, then each local contribution  $(D_1.D_2)_P$  is the dimension of a vectorspace, therefore  $D_1.D_2 \geq 0$ . If equality occurs,  $D_1.D_2 = 0$ , they are disjoint.

The study of divisors on varieties is closely related to the study of sheaves on varieties. If  $D$  is a divisor on a variety  $X$ , then we denote the sheaf

associated to  $D$  as  $\mathcal{O}_X(D)$ . These sheaves are also called *divisorial sheaves*. They are defined in terms of discrete valuations. Let  $D = \sum n_\Gamma \Gamma$  be a divisor on  $X$  and let  $U \subset X$ , then we define

$$\Gamma(U, \mathcal{O}_X(\pm D)) = \{f \in k(X) \mid v_\Gamma(f) \geq \mp n_\Gamma \text{ for all } \Gamma \text{ such that } \Gamma \cap U \neq \emptyset\},$$

where  $k(X)$  is the field of rational functions on  $X$ . For example, the sheaf  $\mathcal{O}_X(-D)$ , means that the sections over an open set  $U$  in  $X$  is just rational functions which are defined on  $U$  and which vanish at every point  $\Gamma$  with multiplicity at least  $n_\Gamma$ . Note that if the  $n_\Gamma \geq 0$  this implies that  $f \in \Gamma(U, \mathcal{O}_X)$ .

When we study curves we are very often dealing with an invariant called the *genus*. There are two (different) kinds of genera, the *arithmetic* genus and the *geometric* genus. The arithmetic genus of a curve  $C$  is defined as  $p_a(C) = h^1(C\mathcal{O}_C)$ . The geometric genus is defined in the following way: If  $C$  is an irreducible projective curve, let  $\pi : C' \rightarrow C$  be its *normalization* or *desingularization*<sup>1</sup>. Then the geometric genus of a curve  $C$ , denoted  $p_g(C)$  or  $g(C)$ , is defined as the arithmetic genus of  $C'$ . Their relation is given in the following proposition.

**Proposition 2.1.** *Let  $C$  be an irreducible curve, possibly singular, on a surface  $S$ , and  $\pi : C' \rightarrow C$  its normalization. Then*

$$p_a(C) = p_g(C') + \sum_{P_i \in \text{Sing}(C)} \delta(P_i),$$

where  $\delta(P_i) > 0$  are numerical invariants of the singularities of  $C$ .

*Proof.* See [Mil96] □

**Remark 2.2.** Note that if  $C$  is a nonsingular curve, then the arithmetic genus and the geometric genus are the same, and the arithmetic genus is always greater than or equal to the geometric genus.

In this thesis we work with smooth curves, so we will write  $g$  for the genus of a curve, unless otherwise specified.

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<sup>1</sup> $\pi : C' \rightarrow C$  is a normalization of the irreducible projective curve  $C$  if  $C'$  is a smooth projective curve and the morphism  $\pi$  is finite and birational.

## 2.2 Linear Systems and Very Ample Divisors

**Definition 2.3.** Let  $D$  be a divisor on a projective variety  $X$ , with  $\mathcal{O}_X(D) \neq 0$ . A *complete linear system*  $|D|$  is the projective space  $\mathbb{P}_*(\mathcal{O}_X(D))$  that parametrizes effective divisors  $D' \geq 0$  linearly equivalent to  $D$ .

For example, consider a special case, where the parameterspace is the projective line,  $\mathbb{P}^1$ . We then call the complete linear system of divisors for a *pencil*. A *net* (resp. *web*) is a special case of a linear system of divisors where the parameterspace is the projective plane (resp. the 3 dimensional projective space).

**Definition 2.4.** A *linear system* on a projective variety  $X$  is a subset of a complete linear system  $|D|$  which is a linear subspace for the projective space structure of  $|D|$ .

This means that a linear system corresponds to the sub-vectorspace of  $H^0(X, \mathcal{O}_X(D))$ . More general, a pencil is a linear system of dimension 1. A net and a web are a linear systems of dimension 2 and 3, respectively.

*Remark 2.5.* Let  $|D|$  is a linear system. We know (cf. [Har77]) that the elements in this linear system is in one-to-one correspondence with the space  $(H^0(X, \mathcal{O}_X(D)) - \{0\})/k^*$ , where  $k$  is as usual an algebraically closed field and  $\mathcal{O}_X(D)$  is the invertible sheaf associated to  $D$ . Very often we are interested in the dimension of a linear system. We see, by the one-to-one correspondance, that

$$\dim_k |D| = \dim_k ((H^0(X, \mathcal{O}_X(D)) - \{0\})/k^*) = h^0(X, \mathcal{O}_X(D)) - 1.$$

**Example 2.6.** let  $|\mathcal{O}_{\mathbb{P}^2}(1)|$  be the linear system of all curves of degree 1 in the projective plane. Then the dimension of this linear system is

$$\dim_k |\mathcal{O}_{\mathbb{P}^2}(1)| = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) - 1 = \binom{1+2}{2} - 1 = 2,$$

so this linear system is a net.

The linear system  $|D|$  may have a subscheme contained in every member of  $|D|$ . We call these subschemes *base points*, and the set of these base points is called a *base locus*, denoted  $\text{Bs}|D|$ . A linear system is *base point free* if it has no base points. For an irreducible curve  $C$ ,  $C^2$  is called its *selfintersection*.

If  $C$  moves in a linear system without fixed components, then  $C \sim C'$ , so we define its selfintersection as  $C^2 = C.C' \geq 0$ . In many cases it happens that the selfintersection is negative,  $C^2 < 0$ .

**Proposition 2.7.** *If  $D_1, D_2 \geq 0$  are effective divisors on a surface  $X$  and  $D_1.D_2 < 0$ , then  $D_1$  and  $D_2$  have at least one common component  $C$ , with  $C^2 < 0$ .*

Divisors on surfaces which provides embeddings and base point free linear systems are fundamental in understanding the geometry on a surface.

**Definition 2.8.** A divisor  $D$  is *nef* on a surface  $X$  if  $D.\Gamma \geq 0$  for every curve  $\Gamma \subset X$ . A divisor  $D$  on a  $K3$  surface  $S$  is *big* if  $D^2 > 0$ .

Linear systems are of special interest, when they provide embeddings into a projective space.

**Definition 2.9.** A line bundle  $\mathcal{O}_X(D)$  on a variety  $X$  for some divisor  $D$  is *very ample* if it is isomorphic to  $\mathcal{O}_X(1)$  for some closed immersion of  $X$  in a projective space. It is *ample* if for any coherent sheaf  $\mathcal{F}$  on  $X$ , the sheaf  $\mathcal{F} \otimes \mathcal{O}_X(D)^{\otimes n}$  is generated by global sections for  $n$  sufficiently large.

For divisors on curves we have the well-known result.

**Proposition 2.10.** ([Har77, Proposition 3.1]) *Let  $D$  be a divisor on a curve  $C$ . Then:*

i) *the linear system  $|D|$  has no base points if and only if for every point  $P \in C$ ,*

$$\dim |D - P| = \dim |D| - 1,$$

ii)  *$D$  is very ample if and only if for every two points  $P, Q \in C$  (including the case  $P = Q$ ),*

$$\dim |D - P - Q| = \dim |D| - 2.$$

Combining this theorem with the Riemann-Roch formula for curves, it is easy to see that any divisor on a curve of high degree is in fact very ample.

**Corollary 2.11.** *If  $C$  is a smooth curve of genus  $g$ , any line bundle of degree  $\geq 2g+1$  on  $C$  is very ample. Moreover, for curves with genus  $g = 0$  or  $g = 1$ , the converse is also true.*

*Proof.* See [Har77, Corollary 3.2].  $\square$

**Theorem 2.12.** (*Hodge Index Theorem*). *Let  $H$  be a divisor on the surface  $X$  with  $H^2 > 0$ , and suppose that  $D$  is a divisor, with  $D.H = 0$  and  $D \not\sim 0$ , then  $D^2 \leq 0$ .*

*Proof.* See ([BPVdV84], IV, 2.15).  $\square$

*Remark 2.13.* Why is the previous theorem called an index theorem? The reason is as follows (cf. [Har77]):

Let  $\text{Pic}^n(X)$  be the subgroup of divisor classes which are numerically equivalent to zero, and let  $\text{Num}(X) = \text{Pic}(X)/\text{Pic}^n(X)$ . Then the intersection pairing induces a nondegenerate bilinear mapping

$$\text{Num}(X) \times \text{Num}(X) \rightarrow \mathbb{Z}.$$

The Nèron-Severi Theorem says that the group of divisors modulo algebraic equivalence is a finitely generated abelian group. Since  $\text{Num}(X)$  is a quotient group of the Nèron-Severi group, it is finitely generated, and therefore free, since it is torsion free. We can consider the vector space  $\text{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  over  $\mathbb{R}$ , and the induced bilinear form. In the paper of J. J. Sylvester [Syl89], it is shown that such a bilinear form can be diagonalized with  $\pm 1$ 's on the diagonal. Moreover, it is also shown that the number of  $+1$ 's and the number of  $-1$ 's are invariant of the bilinear form. The difference of these two numbers called *signature* or *index* of the bilinear form. So Theorem 2.12 is called an index theorem, because the diagonalized intersection pairing has only one  $+1$ , which correspond to a real multiple of  $H$ , and all the rest are  $-1$ 's.

There is a consequence (cf. [Mil96, Chapter 3]) of the Hodge Index theorem, which can be very useful in calculations. If  $D$  and  $C$  are divisors and  $(\lambda C + \gamma D)^2 > 0$  for some real numbers  $\lambda, \gamma \in \mathbb{R}$ , the determinant

$$\det \begin{vmatrix} C^2 & D.C \\ C.D & D^2 \end{vmatrix} \leq 0,$$

with equality if and only if nonzero rational linear combination is numerical linear equivalent to zero, i.e.  $\alpha C + \beta D \sim 0$ , with  $\alpha, \beta \in \mathbb{Q}$ .

## 2.3 Two Vanishing Theorems

In this section we state two vanishing theorems, which we will use in Chapter 6 and Chapter 7.

**Theorem 2.14.** (*Kodaira Vanishing*) Let  $H$  be an ample divisor on a nonsingular  $n$ -fold  $X$  over a field of characteristic zero, then  $H^i(X, \mathcal{O}_X(K_X + H)) = 0$  for all  $i > 0$ .

If the assumptions in the Kodaira Vanishing theorem are satisfied, then one can observe that the dual  $H^j(X, \mathcal{O}_X(-H)) = 0$  for all  $j < n$ . Another vanishing theorem, is that of Kawamata-Viehweg.

**Theorem 2.15.** (*Kawamata-Viehweg Vanishing*) Let  $\mathcal{L}$  be a big and nef line bundle on a projective surface  $X$  with canonical divisor  $K_X$ , then  $H^i(X, \mathcal{O}_X(\mathcal{L} + K_X)) = 0$  for all  $i \geq 1$ .

## 2.4 Important Exact Sequences

When we study geometry on varieties, sheaf cohomology is a very important tool and in this section we will state some standard exact sequences we will use frequently. Let  $X$  be a nonsingular  $n$ -fold and  $Y$  a codimension 1 subvariety  $Y \subset X$ . Then we have a natural short exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0, \quad (2.4.1)$$

where the ideal sheaf  $\mathcal{I}_Y = \mathcal{I}_{Y/X} = \mathcal{O}_X(-Y)$ , is a line bundle on  $X$ . The restriction of this sheaf is called the *conormal bundle*  $(\mathcal{N}_{Y/X})^\vee$  on  $Y$ , which can be written in many ways:

$$(\mathcal{N}_{Y/X})^\vee = \mathcal{I}_Y / \mathcal{I}_Y^2 = \mathcal{I}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \mathcal{O}_Y(-Y).$$

If we tensor the exact sequence (2.4.1) with the sheaf  $\mathcal{O}_X(Y)$  we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(Y) \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0, \quad (2.4.2)$$

where  $\mathcal{N}_{Y/X} = \mathcal{O}_Y(Y) = \mathcal{O}_X(Y) \otimes \mathcal{O}_Y$  is the normal bundle, which is a vector bundle of rank  $r = \text{codim}(Y, X)$ . If we have the situation,  $Y \subseteq X \subseteq \mathbb{P}^r$ , we have the following normal bundle sequence

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/\mathbb{P}^r} \longrightarrow \mathcal{N}_{X/\mathbb{P}^r} \otimes \mathcal{O}_Y \longrightarrow 0. \quad (2.4.3)$$

The following theorem is due to P. Griffiths and J. Harris in [GH83, p. 252] and will be used to prove Proposition 7.4 in Chapter 7;

**Theorem 2.16.** *If  $C$  is a smooth curve on a smooth surface  $X \subseteq P^3$ , then the bundle sequence (2.4.3) splits if and only if  $C$  is a complete intersection with  $X$ .*

The third exact sequence is

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \otimes \mathcal{O}_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0, \quad (2.4.4)$$

where  $\mathcal{T}_Y = (\Omega_{Y/k})^\vee$  called the tangent bundle<sup>2</sup>. We also have a very important formula, called the *adjunction formula*,

$$K_Y = (K_X + Y)|_Y.$$

**Proposition 2.17.** *If  $C$  is a nonsingular curve of genus  $g$  on the surface  $X$ , and if  $K_X$  is the canonical divisor on  $X$ , then*

$$2p_a(C) - 2 = C \cdot (C + K_X). \quad (2.4.5)$$

*Proof.* See [Har77, V, Proposition 1.5]. □

*Remark 2.18.* When  $C$  is an irreducible smooth curve on a  $K3$  surface  $S$ , it is very easy to calculate its selfintersection. It is just  $2p_a(C) - 2$ , since  $K_S$  is trivial.

**Example 2.19.** Let  $C$  be a twisted quartic curve in  $\mathbb{P}^3$ . We want to calculate its genus. Then by the adjunction formula (2.4.5), we have

$$2g - 2 = d(d - 4).$$

Since  $C$  is a curve of degree 4, this gives that  $g = 1$ , which is an elliptic curve.

**Example 2.20.** Let  $S$  be a  $K3$  surface. If

- i)  $C \subset S$  is a rational curve, then  $C^2 = -2$ .
- ii)  $C \subset S$  is an elliptic curve, then  $C^2 = 0$ .

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<sup>2</sup>Notice that this is the dual of the canonical bundle on  $Y$ .





# Chapter 3

## Geometry on $K3$ Surfaces

In this chapter we will look at some basic properties on  $K3$  surfaces and we will look at three very important invariants of curves. These invariants will be important for us in Chapter 5, when we study curves in  $\mathbb{P}^r$  as  $K3$  sections. In the last section we will look at some examples. Some of these examples will be studied more closely in Chapter 7.

### 3.1 Basic Properties

A variety  $X \subset \mathbb{P}^r$  is *linearly normal* or *embedded by a complete linear system* if  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \longrightarrow H^0(X, \mathcal{O}_X(1))$  is surjective. This means that  $X$  is not the linear projection of a variety spanning a higher dimensional  $X \subset \mathbb{P}^{r+1}$ . The following theorem shows an interesting property of  $K3$  surfaces, i.e. all smooth hyperplane sections are canonical curves.  $K3$  surfaces are the only family of surfaces that satisfy this property. The following result can be found in Section 3.3 in [Mil96].

**Theorem 3.1.** *Let  $S \subset \mathbb{P}^r$  be a nonsingular surface. Then  $S$  is a  $K3$  surface embedded by a complete linear system if and only if one (every) nonsingular hyperplane section is a canonical curve.*

The next classical theorem is known for all algebraic geometers, and you can find it almost all books in algebraic geometry. There is a more generalized version of this theorem, which is known as Hirzebruch-Riemann-Roch (see for example in [Har77, Appendix A]).

**Theorem 3.2.** (*Riemann-Roch*) *If  $D$  is any divisor on a surface  $X$ , then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D - K_X) + \chi(\mathcal{O}_X). \quad (3.1.1)$$

*Proof.* See for example [Mil96] or [Har77].  $\square$

Using the vanishing theorem of Grothendieck ([Har77, Theorem 2.7]), the criteria for  $K3$  surfaces and Serre duality, the Riemann-Roch formula (3.1.1) becomes very simple,

$$h^0(\mathcal{O}_S(D)) - h^1(\mathcal{O}_S(D)) + h^0(\mathcal{O}_S(-D)) = \frac{1}{2}D^2 + 2 \quad (3.1.2)$$

**Corollary 3.3.** ([Mil96, Corollary 3.7.1]) *Let  $D$  be a divisor on a  $K3$  surface  $S$ . Then the following holds:*

- i) *If  $D^2 \geq -2$ , then  $H^0(\mathcal{O}_S(D)) \neq 0$  or  $H^0(\mathcal{O}_S(-D)) \neq 0$ .*
- ii) *If  $D^2 \geq 0$ , then  $D \sim 0$ , or  $h^0(\mathcal{O}_S(D)) \geq 2$ , or  $h^0(\mathcal{O}_S(-D)) \geq 2$ .*
- iii) *If  $D$  is an effective divisor on  $S$  with  $h^0(\mathcal{O}_S(D)) = 1$ , then  $D'^2 \leq -2$  for every divisor  $D'$  with  $0 < D' \leq D$ , and in particular  $D$  is a sum of  $-2$ -curves with  $D^2 \leq -2$ .*

**Proposition 3.4.** *Let  $E_1$  and  $E_2$  be divisors on a  $K3$  surface  $S$ , and suppose  $E_1^2 > 0$ . Then*

$$E_1^2.E_2^2 \leq (E_1.E_2)^2 \text{ with equality if and only if } (E_1.E_2)E_1 \sim E_1^2E_2$$

*Proof.* This follows from the Hodge Index Theorem above and [Fri98, Chapter 1, Exercise 10] and using that numeric equivalence and linear equivalence are the same for divisors on a  $K3$  surface.  $\square$

## 3.2 Three Important Invariants

In this section we will discuss three invariants of curves. These invariants will give us information about the curves on  $K3$  surfaces later on.

Let  $C$  be a smooth irreducible curve of genus  $g \geq 2$ . We denote by  $\mathfrak{g}_d^r$  a linear system on  $C$  of dimension  $r$  and degree  $d$ .

First we will say something about the motivation for defining the Clifford index and the gonality. It is natural to start with Clifford's theorem. If  $D$  is a nonspecial divisor on a curve, then we can find  $\dim|C|$  exactly as a function of the degree of  $D$  by the Riemann-Roch theorem for curves. But when  $D$  is a special divisor,  $\dim|D|$  does not depend only on the degree of  $D$ . Therefore it is useful to have a bound on  $\dim|D|$ , and Clifford's theorem gives us the answer.

**Theorem 3.5.** (*Clifford's theorem*) *Let  $D$  be an effective special divisor on the curve  $C$ . Then*

$$\dim|D| \leq \frac{1}{2} \deg D. \quad (3.2.1)$$

*Furthermore, equality occurs if and only if either  $D = 0$  or  $D = K_C$  or  $C$  is hyperelliptic and  $D$  is multiple of the unique  $\mathfrak{g}_2^1$  on  $C$ .*

*Proof.* See [Har77, Theorem 5.4] or [SD73, Theorem 1.4].  $\square$

Using elementary algebraic manipulations of (3.2.1) and the fact that  $\dim|C| = h^0(\mathcal{O}_C - 1)$ , the formula for the Clifford index is clear.

**Definition 3.6.** Let  $C$  be a curve. Then we define the *Clifford index* of a smooth irreducible curve  $C$  of genus  $g \geq 4$  as

$$\text{Cliff}(C) := \min\{\text{Cliff}(D) \mid D \in \text{Div}(C), h^0(\mathcal{O}_C(D)) \geq 2, h^1(\mathcal{O}_C(D)) \geq 2\},$$

where for any divisor  $D$  on  $C$ , we have

$$\text{Cliff}(D) = \deg(D) - 2(h^0(\mathcal{O}_C(D)) - 1) = g + 1 - h^0(\mathcal{O}_C(D)) - h^1(\mathcal{O}_C(D)).$$

Moreover, if  $D$  is a divisor on  $C$  satisfying  $h^0(\mathcal{O}_C(D)) \geq 2$  and  $h^1(\mathcal{O}_C(D)) \geq 2$ , then one says that  $D$  contributes to the Clifford index of  $C$ . In addition, one says that  $D$  computes the Clifford index if  $\text{Cliff}(C) = \text{Cliff}(\mathcal{O}_C(D))$ .

Notice that the two formulas for the Clifford index for the divisor  $D$  are related by the Riemann-Roch theorem.

*Remark 3.7.* If the genus of a curve is less than 4, then there are no line bundles  $\mathcal{O}_C(D)$  with  $h^0(\mathcal{O}_C(D)) \geq 2$  and  $h^1(\mathcal{O}_C(D)) \geq 2$ . In this situation we say that a nonhyperelliptic curve of genus 3 has Clifford index 1, while any hyperelliptic curve of genus  $\leq 3$  has Clifford index 0. Thus, by Clifford's theorem,  $\text{Cliff}(C) \geq 0$  and  $\text{Cliff}(C) = 0$  if and only if  $C$  is hyperelliptic or  $g \leq 1$ .

It is known from Brill-Noether Theory that a general curve of genus  $g$  possesses a line bundle  $\mathcal{L}$  with  $h^0(\mathcal{L}) \geq h^0$  and  $h^1(\mathcal{L}) > h^1$  if and only if  $h^0 h^1 \leq g$ , [ACGH]. Then it follows that a general curve has Clifford index  $\lfloor (g-1)/2 \rfloor$ . Geometrically, the Clifford index measures the expectation of non-trivial divisors on a curve, in other words, it measures how far a curve is from being hyperelliptic. The smaller  $\text{Cliff}(C)$  is, the better chance we can find divisors which have many sections for their degree. By Clifford's theorem and the existence theorem of special divisors ([LK72], [Kem71] or [GH80]),  $\text{Cliff}(C)$  is bounded by  $0 \leq \text{Cliff}(C) \leq \frac{g-1}{2}$ . By Serre duality, we also see that

$$\text{Cliff}(\mathcal{O}_C(D)) = \text{Cliff}(\mathcal{O}_C(-D) \otimes \omega_C).$$

**Example 3.8.** Let  $D$  be divisor on a curve  $C$  such that the line bundle  $\mathcal{O}_C(D)$  is nonspecial, that is,  $h^1(\mathcal{O}_C(D)) = 0$ . Then

$$\begin{aligned} \text{Cliff}(D) &= \deg(D) - 2(h^0(\mathcal{O}_C(D)) - 1) = \deg(D) - 2(\deg(D) + 1 - g - 1) \\ &= 2g - \deg(D). \end{aligned}$$

We see that the Clifford index only depends on the degree of  $\mathcal{O}_C(D)$ , as we expected by our discussion of nonspecial divisors just before Clifford's Theorem above.

**Definition 3.9.** We define the *gonality*<sup>1</sup> of a curve  $C$  as

$$\text{gon}(C) := \min\{k \in \mathbb{Z}_+ \mid C \text{ has a } \mathfrak{g}_k^1, \text{ but no } \mathfrak{g}_{k-1}^1\}.$$

The gonality of a curve measures how rational a curve is, that is, the minimum degree of the map  $\phi : C \rightarrow \mathbb{P}^1$ . An upper bound on the Clifford index is

$$\begin{aligned} \text{Cliff}(C) &= \min\{\text{Cliff}(D)\} \\ &= \min\{\deg(D) - 2(h^0(\mathcal{O}_C(D)) - 1)\} \\ &\leq \text{gon}(C) - 2(h^0(\mathcal{O}_C(D)) - 1) \\ &\leq \text{gon}(C) - 2, \end{aligned}$$

since  $h^0(\mathcal{O}_C(D)) \geq 2$ , by the definition of Clifford index. On the other hand, M. Coppens and G. Martens showed in the paper ([CM91]) that

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<sup>1</sup>The name gonality, comes from the habit of calling a curve with a tree-to-one map to  $\mathbb{P}^1$  "trigonal".

$\text{Cliff}(C) \geq \text{gon}(C) - 3$ . This means that there is a close relationship between these two invariants. This is not surprising, because rational curves are not that different from hyperelliptic curves. Curves that satisfies the equality,  $\text{gon}(C) = \text{Cliff}(C) + 3$ , are called *exceptional curves*.

It is known that for a fixed genus, curves of any possible gonality occur. Ballico proved [Bal86] that this also holds for the Clifford index.

While the gonality tells us something about the minimal degree of a  $\mathfrak{g}_k^1$ , it is natural to define a measure of the minimal dimension of such a linear system.

**Definition 3.10.** The third invariant of a curve is the *Clifford dimension* of  $C$  defined as

$$\text{Cliff-dim}(C) := \min\{r \geq 1 \mid \exists \mathfrak{g}_k^r \text{ on } C \text{ with } k \leq g - 1, \text{ such that } k - 2r = \text{Cliff}(C)\}.$$

From this definition, it is easy to see that a curve  $C$  is  $(\text{Cliff}(C) + 2)$ -gonal if and only if its Clifford dimension is 1. Therefore, the exceptional curves have Clifford dimension  $\geq 2$ .

**Example 3.11.** It is difficult to construct examples of exceptional curves. Here is an example of curves with Clifford dimension 2 and 3.

- i) Smooth plane curves of degree  $d \geq 5$  are precisely the curves of Clifford dimension 2. Moreover, any smooth plane curve has gonality  $d - 1$ , and since  $\deg(\mathcal{O}_C(1)) = d$  and  $h^0(C, \mathcal{O}_C(1)) = 3$ , we see that  $\text{Cliff}(C) = d - 4 = \text{gon}(C) - 3$ . We only have to check whether  $h^1(C, \mathcal{O}_C(1)) \geq 2$ . By the Riemann-Roch theorem and that the genus of plane curves are given by

$$g = \frac{(d-1)(d-2)}{2},$$

we get

$$\begin{aligned} h^0(C, \mathcal{O}_C(1)) &= 3 = d + 1 - g(C) + h^1(C, \mathcal{O}_C(1)) \\ &= d + 1 - \frac{1}{2}(d-1)(d-2) + h^1(C, \mathcal{O}_C(1)). \end{aligned}$$

This implies that  $h^1(C, \mathcal{O}_C(1)) = \frac{1}{2}d^2 - \frac{5}{2}d + 3$ , which is exactly  $\geq 2$  when  $d \geq 5$ . The case  $d = 1$  we discard. The converse follows from [ELMS89, Lemma 1.1], stating that any divisor  $A$  computing the Clifford dimension is very ample if  $h^0(C, \mathcal{O}_C(A)) \geq 3$ .

- ii) Curves of Clifford dimension 3 occur only in genus 10 as complete intersections of two cubic surfaces in  $\mathbb{P}^3$ .

It is conjectured [ELMS89] that curves of Clifford dimension  $\geq 3$  are very rare. In fact, the statement says that there is only one pair  $(d, g) = (4r - 3, 4r - 2)$  for any given Clifford dimension  $\geq 3$ , where  $g = g(C)$  and  $d = \deg(A)$  where  $A$  computes  $\text{Cliff}(C)$ .

In the next section we will look at some well-known conjectures concerning exceptional curves, the Clifford index and gonality, and we will see how they have been tested on  $K3$  surfaces.

### 3.3 Conjectures

Let us recall the well-known conjecture in [ELMS89].

**Conjecture 3.12.** (*Eisenbud, Lange, Martens and Shreyer*) *Let  $C$  be a smooth curve of Clifford dimension  $r \geq 3$ . Then:*

- (a)  *$C$  has a unique genus  $g = 4r - 2$  and Clifford index  $c = 2r - 3$ ;*
- (b)  *$C$  has a unique line bundle  $A$  computing  $c$  (and  $\deg A = g - 1$ );*
- (c)  *$A^2 \simeq \omega_C$  and  $A$  embeds  $C$  as an arithmetically Cohen-Macaulay curve in  $\mathbb{P}^r$ ;*
- (d)  *$C$  is  $2r$ -gonal, and there is one-dimensional family of pencils of degree  $2r$ , all of the form  $|A - B|$ , where  $B$  is a divisor of  $2r - 3$  point of  $C$ .*

The conjecture above is known as the ELMS conjecture. In the paper [ELMS89], the conjecture is proved for  $r \leq 9$ , and in general it is proved that if  $C$  satisfies (a), then it also satisfies (b)-(d). In the same paper ([ELMS89, Theorem 4.3]) the authors constructed an infinite series of examples of exceptional curves lying on  $K3$  surfaces, which are known as the ELMS examples. The interesting thing is that the line bundles in these cases are not ample. Knutsen [Knu09] made the following generalization of these examples.

**Example 3.13.** (**The Generalized ELMS Examples**) Let  $L$  be a line bundle on a  $K3$  surface  $S$  such that  $L \sim 2D + \Gamma$  with  $D$  and  $\Gamma$  smooth curves satisfying  $D^2 \geq 2$ ,  $\Gamma^2 = -2$  and  $\Gamma.D = 1$ . Assume furthermore that there is no line bundle  $B$  on  $S$  satisfying  $0 \leq B^2 \leq D^2 - 1$  and  $0 < B.L - B^2 \leq D^2$ .

Then  $|L|$  is base point free and all smooth curves in  $|L|$  are exceptional, of genus  $g = 2D^2 + 2 \geq 6$ , Clifford index  $c = D^2 - 1 = \frac{g-4}{2}$  and Clifford dimension  $r = \frac{1}{2}D^2 + 1$ . Moreover, for any smooth curve  $C \in |L|$  the Clifford index is computed only by  $\mathcal{O}_C(D)$ .

The curves in the generalized ELMS examples have  $\dim W_k^1(C) = 1$  and  $\rho(g, k, 1) = 0$ , where  $k = \text{gon}(C)$ ,  $W_k^1(C)$  is the scheme which parametrizes line bundles  $D \in \text{Pic}^1(C)$  with  $h^0(C, \mathcal{O}_C(D)) \geq 2$  and  $\rho(g, k, 1) = g - 2(g - k + 1)$  is a number which occurs in Brill-Noether theory<sup>2</sup> and in this case ( $r = 1$ ), the Brill-Noether Theorem (cf. [ACH85]) says that when this number is negative, the general curve has no pencils.

For many years, mathematicians have tried to find out whether exceptional linear systems on a curve on certain surface propagate to the members of  $|C|$ . For  $K3$  surfaces, this has been a hot topic. Here is a short résumé:

Saint-Donat proved [SD74] that  $C$  possesses a  $\mathfrak{g}_2^1$  and a  $\mathfrak{g}_3^1$  if and only if every smooth curve in  $|C|$  does. Miles Reid [Rei76] extended this result to  $\mathfrak{g}_d^1$ s. Harris and Mumford conjectured, which is unpublished, that the gonality of linearly equivalent curves does not change. In 1989 this was proven to be false, by Donagi and Morrison [DM89]. They constructed the following famous counterexample.

**Example 3.14.** /**The Morrison-Donagi example**) Let  $\pi : S \rightarrow \mathbb{P}^2$  be a  $K3$  surface of genus 2 which is a double cover of  $\mathbb{P}^2$  branched along a smooth sextic, and let  $L := \pi^*(\mathcal{O}_{\mathbb{P}^2}(3))$ . The arithmetic genus of the curve in  $|L|$  is 10. The smooth curves in the codimension one linear subspace  $|\pi^*(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)))| \subset |L|$  are bielliptic, whence with gonality 4. On the other hand the general curve in  $|L|$  is isomorphic to a smooth plane sextic and therefore has gonality 5.

*Remark 3.15.* Knutsen proved ([Knu09, Theorem 1.2]) that when  $C$  is an exceptional curve on a  $K3$  surface  $S$ , then  $C$  is either a smooth plane sextic belonging to the Donagi-Morrison example or the line bundle  $\mathcal{O}_S(C)$  as in the generalized ELMS examples (Example 3.13). In particular,  $C$  satisfies Conjecture 3.12.

Few years earlier, the Harris-Mumford conjecture was modified by Green [Gre84b]. Now the conjecture says that all smooth curves in the same linear system have the same Clifford index. This was proved in the paper [GL87]

<sup>2</sup>We will say more about Brill-Noether theory in Section 5.1.

by Green and Lazarsfeld. Later on, Ciliberto and Pareschi proved [CP95] that this is the only counterexample when the line bundle  $\mathcal{O}_S(C)$  is ample. For  $K3$  surfaces, the story ends in the paper of Knutsen [Knu09], who proved the following theorem:

**Theorem 3.16.** *Let  $S$  be a  $K3$  surface and  $L$  a globally generated line bundle on  $S$ . If the gonality of the smooth curves in  $|L|$  is not constant, then  $S$  and  $|L|$  are as in the Donagi-Morrison example.*

**Example 3.17.** Here are some examples of curves with known gonality.

- i) The gonality is 1 for curves with genus  $g = 0$ , i.e. the rational curves.
- ii) Curves with gonality 2 are hyperelliptic curves, including elliptic curves. Hyperelliptic curves are for example birational to  $y^2 = f_{2g+2}(x)$  (see [Mil96, p.65]), or equivalently it has a divisor class  $D$  with  $\deg D = 2$  and  $\dim |D| = 1$ , therefore a  $\mathfrak{g}_2^1$ .
- iii) The gonality of a generic curve is the floor function  $\lfloor \frac{g+3}{2} \rfloor$ .

For curves with genus 0, 1 or 2, the gonality is completely determined by the curve's genus. For curves with higher genera ( $\geq 3$ ), this is not the case and one must come up with other techniques to find it. We know how to compute the gonality of curves attaining the Castelnuovo bound (see [ACH85]) and the gonality of complete intersections in  $\mathbb{P}^3$  (cf. [Bas96]):

**Example 3.18.** If  $C \subseteq \mathbb{P}^3$  is a smooth complete intersection of type  $(a, b)$ , then  $\text{gon}(C) = ab - l$ , where  $l$  is the degree of a maximal linear divisor on  $C$ .

Green and Lazarsfeld [GL85] published a conjecture about the gonality of curves which are embedded of sufficiently high degree.

**Conjecture 3.19. (*The Gonality Conjecture*)** *For any smooth curve  $C$  of gonality  $k$ , every nonspecial globally generated line bundle  $L$  on  $C$  of sufficiently high degree satisfies*

$$K_{h^0(L)-k,1}(C, L) = 0,$$

where  $K_{h^0(L)-k,1}(C, L)$  denotes the  $(h^0(L) - k, 1)$ -th Koszul cohomology group of the line bundle  $L$ .



This conjecture is now proven by Aprodu and Voisin [AV03a] for generic curves with of genus  $g$  and gonality  $k$ , if  $g/3 < k < [g/2] + 2$ , and in some further cases by the same authors [AV03b].

Another interesting conjecture is Green's Conjecture. It predicts that one can read off special linear series on an algebraic curve, by looking at the syzygies of its canonical embedding. Green formulated this conjecture in [Gre84a].

**Conjecture 3.20. (*Green's Conjecture*)** *If  $C$  is a smooth algebraic curve of genus  $g$ ,  $K_{i,j}(C, K_C)$  denotes the  $(i, j)$ -th Koszul cohomology group of the canonical bundle  $K_C$ , then*

$$K_{p,2}(C, K_C) = 0, \text{ for all } p < \text{Cliff}(C).$$

Many attempts have been made to settle this question, and some nice results have been obtained ([Voi88] and [Sch86]). For  $K3$  surfaces, C. Voisin ([Voi02] and [Voi03]) achieved a major breakthrough by showing that Green's Conjecture holds for smooth curves  $C$  lying on  $K3$  surfaces  $S$  with  $\text{Pic}(S) = \mathbb{Z}C$ . Using Voisin's work, as well as a degenerate form of [HR98], it has been proved [Apr05] that Green's conjecture holds for any curve  $C$  of genus  $g$  and gonality  $\text{gon}(C) = k \leq \frac{g+2}{2}$ , that satisfies the *linear growth condition*

$$\dim W_{k+n}^1 \leq n, \text{ for } 0 \leq n \leq g - 2k + 2.$$

In the paper of M. Aprodu and G. Farkas [AF11], we get a complete solution<sup>3</sup> to the Green's Conjecture for smooth curves on arbitrary  $K3$  surfaces. They proved the following result;

**Theorem 3.21.** *Green's Conjecture holds for every smooth curve  $C$  lying on an arbitrary  $K3$  surface  $S$ .*

In general, both Green's Conjecture and the Gonality Conjecture are still open problems today.

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<sup>3</sup>Completely independent, Nils Henry Rasmussen proved the Green's Conjecture on  $K3$  surfaces in his PhD thesis almost at same time as Farkas and Aprodu. The proof was not published, because it is basically the same as in [AF11].

### 3.4 Examples

**Example 3.22.** We want to find the canonical sheaf of  $\mathbb{P}^r$ . From [Har77, Chapter II, Theorem 8.13] we have an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^r/k} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-1)^{r+1} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow 0. \quad (3.4.1)$$

Applying the exterior power  $\wedge^r \Omega_{\mathbb{P}^r/k}$  to the exact sequence (3.4.1). We get  $\omega_{\mathbb{P}^r} \cong \mathcal{O}_{\mathbb{P}^r}(-r-1)$ .

We have already seen an example of a  $K3$  surface, namely,  $S_2 \rightarrow \mathbb{P}^2$ , the double covering of  $\mathbb{P}^2$  branched along a smooth plane sextic<sup>4</sup>.

The first examples of  $K3$  surfaces are the ones which are complete intersections. One can show (see [Har77, Exercises II, 8.4 and III, 5.5] or [Mil96, Chapter 3, Exercises 5 and 6]) that there are exactly three types of  $K3$  complete intersections. In the following examples we take a look at these.

**Example 3.23.** (A hyperquartic in  $\mathbb{P}^3$ ) Let  $S = S_4 \subset \mathbb{P}^3$  be a nonsingular quartic surface. By using Exercise II, 8.4 (e) in [Har77], we have that the canonical sheaf on  $S$  is  $\omega_S \cong \mathcal{O}_S$ . This means that any trivial divisor  $K_S$  in linear equivalence class corresponding to the canonical sheaf is zero. Moreover, consider the exact sequence (2.4.1)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

From the long exact sequence of cohomology we have

$$\dots \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(\mathcal{O}_S) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^3}(-4)) \longrightarrow \dots$$

Now  $H^2(\mathcal{O}_{\mathbb{P}^3}(-4)) = 0$  and  $H^1(\mathcal{O}_{\mathbb{P}^3}) = 0$ , by [Har77, Theorem 5.1]. Therefore  $S$  is a  $K3$  surface.

**Example 3.24.** (Ci of a hyperquadric and a hypercubic in  $\mathbb{P}^4$ ) If we look at the complete intersection  $S = S_{2,3} \subset \mathbb{P}^4$ . We find that the canonical sheaf  $\omega_S \cong \mathcal{O}_S$ , again by Exercise II, 8.4 (e) in [Har77], hence  $K_S = 0$ . By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

we obtain that  $H^1(S, \mathcal{O}_S) = 0$ . Therefore  $S$  is a  $K3$  surface.

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<sup>4</sup>See the Donagi-Morrison example on page 31.

The same argument can be used to show that the complete intersection of three hyperquadrics in  $\mathbb{P}^5$ ,  $S_{2,2,2} \subset \mathbb{P}^5$ , is a  $K3$  surface.

Until now, we have only looked at  $K3$  surfaces living in  $\mathbb{P}^2$ ,  $\mathbb{P}^3$ ,  $\mathbb{P}^4$  and  $\mathbb{P}^5$ .  $K3$  surfaces which live in  $\mathbb{P}^6$  and  $\mathbb{P}^7$  are of a much more complicated structure. First we define:

- i) For a vector space  $V^d$  of dimension  $d$ , we write  $G(r, V^d)$  for the Grassmann variety of  $r$ -dimensional subspaces of  $V$ .
- ii) The variety  $\sum_{12}^{10} \subseteq \mathbb{P}^{15}$  is a 10-dimensional spinor variety<sup>5</sup> of degree 12. In addition, if  $V^{10}$  is a 10-dimensional vector space with a nondegenerate second symmetric tensor  $\lambda$ . The  $\sum_{12}^{10}$  is one of the two components of the subset of  $G(5, V^{10})$  consisting of 5-dimensional totally isotropic<sup>6</sup> quotient spaces.

Then the surfaces  $(1, 1, 1, 2) \cap G(2, V^5) \subseteq \mathbb{P}^6$  and  $(1^8) \cap \sum_{12}^{10} \subset \mathbb{P}^7$  are both  $K3$  surfaces. Shigeru Mukai classified projective models of polarized<sup>7</sup>  $K3$  surfaces of low genera<sup>8</sup>.

We also have an example of a  $K3$  surface of degree 4, which is a product of two elliptic curves. E. Kummer constructed these surfaces in the 1860s.

**Example 3.25.** (cf. [SI77]) Let  $A$  be an abelian surface which is the product of two smooth elliptic curves  $C_1$  and  $C_2$ . A *Kummer surface*  $S = \text{Km}(A)$  is the minimal non-singular model of the quotient surface  $A/\tau_A$  of  $A$  by the inverse automorphism  $\tau_A$  ( $\tau_A(z) = -z$ ), which has the 16 singular points corresponding to the points of order 2 of  $A$ . Let  $u_i$  (or  $u'_j$ ) ( $1 \leq i \leq 4$ ) be the 4 points of order 2 on the elliptic curve  $C_1$  (or  $C_2$ ), we denote by  $E_{ij}$  ( $1 \leq i, j \leq 4$ ) the non-singular rational curve on  $S$  corresponding to the point  $(u_i, u'_j)$  of  $A$ . Moreover, we let  $F_i$  (or  $G_i$ ) be the non-singular rational curve on  $S$ , which is the image of  $u_i \times C_2$  (or  $C_1 \times u'_j$ ) under the natural rational map  $\alpha : A \rightarrow S$ .

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<sup>5</sup>A *Spinor variety* is a projective variety, which is the set of all maximal totally isotropic vector subspaces to a  $2r$ -dimensional vectorspace  $V$ , [Ang11].

<sup>6</sup>A quotient  $f : V \rightarrow V'$  is *totally isotropic* with respect to  $\lambda$  if  $f \otimes f(\lambda)$  is zero on  $V' \otimes V$ .

<sup>7</sup>A pair  $(S, L)$  of a  $K3$  surface  $S$  and a base point free line bundle  $L$  with  $L^2 = 2g - 2$  is called a *polarized  $K3$  surface* of genus  $g$ .

<sup>8</sup>See [Muk95] or [JK04] for more classification of  $K3$  surfaces.

The Kummer surface  $S$  has two elliptic pencils  $\Psi_n : S \rightarrow \mathbb{P}^1$ , which are induced by the projections  $A \rightarrow C_n$  ( $n = 1, 2$ ). Each  $\Psi_n$  has 4 singular fibres:

$$2F_i + \sum_{j=1}^4 E_{ij} \sim F \quad (1 \leq i \leq 4),$$

$$2G_j + \sum_{i=1}^4 E_{ij} \sim G \quad (1 \leq j \leq 4),$$

where  $F$  (or  $G$ ) is a general fiber of  $\Psi_1$  (or  $\Psi_2$ ). The intersection numbers between these curves are given as follows:

$$FG = 2, FE_{ij} = GE_{ij} = F_i G_j = 0, FG_j = GF_i = 1,$$

$$E_{ij}^2 = F_i^2 = G_j^2 = -2, F_k E_{ij} = \delta_{ki}, G_k E_{ij} = \delta_{kj}.$$

The configuration formed by the rational curves  $E_{ij}, F_i$  and  $G_j$  is called the *double Kummer pencil* on  $S = \text{Km}(C_1 \times C_2)$ .

# Chapter 4

## Useful Results on $K3$ Surfaces

In this chapter we have collected some results we will use in the next chapters, only Corollary 4.9 is new.

### 4.1 Existence Results

We will need the following theorem in Section 6.1, where we will show existence of an algebraic  $K3$  surface with Picard rank 3.

**Proposition 4.1.** *If  $\rho \leq 10$ , then every even lattice of signature  $(1, \rho - 1)$  occurs as the Nèron-Severi group of some algebraic  $K3$  surface.*

*Proof.* See [Mor84, Theorem 2.9 i)] or [Nik80].  $\square$

*Remark 4.2.* Another approach to show existence of a  $K3$  surface is to deform one with Picard group of high rank to another one with Picard group of low rank, using deformation theory [MM83].

The next theorem shows completely under what conditions on  $d, g$  and  $r$  there exists a projective  $K3$  surface with a smooth curve  $C$  with degree  $d$  and genus  $g$  on it.

**Theorem 4.3.** *([Knu02, Theorem 1.1]) Let  $r \geq 3$ ,  $d > 0$  and  $g \geq 0$  be integers. Then there exists a  $K3$  surface  $S$  of degree  $2r - 2$  in  $\mathbb{P}^r$  containing a smooth curve  $C$  of degree  $d$  and genus  $g$  if and only if*

- i)  $g = d^2/4(r-1)+1$  and there exists integers  $k, m \geq 1$  and  $(k, m) \neq (2, 1)$  such that  $r - 1 = k^2m$  and  $2r - 2$  divides  $kd$ ,*

ii)  $d^2/4(r-1) < g < d^2/4(r-1) + 1$  except in the following cases,

- a)  $d \equiv \pm 1, \pm 2 \pmod{2r-2}$ ,
- b)  $d^2 - 4(r-1)(g-1) = 1$  and  $d \equiv r-1 \pm 1 \pmod{2r-2}$ ,
- c)  $d^2 - 4(r-1)(g-1) = r-1$  and  $d \equiv r-1 \pmod{2r-2}$ ,
- d)  $d^2 - 4(r-1)(g-1) = 1$  and  $d-1$  or  $d+1$  divides  $2r-2$ .

iii)  $g = d^2/4(r-1)$  and  $d$  is not divisible by  $2r-2$ ,

iv)  $g < d^2/4(r-1)$  and  $(d, g) \neq (2r-1, r)$ .

Furthermore, in case i)  $S$  can be chosen such that  $\text{Pic}(S) = \mathbb{Z}_{\frac{2r-2}{dk}}^{2r-2} C = \mathbb{Z}_{\frac{1}{k}} H$  and in cases ii)-iv) such that  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ , where  $H$  is a hyperplane section of  $S$ .

If  $r \geq 5$ ,  $S$  can be chosen to be scheme-theoretically an intersection of quadrics in cases i), iii) and iv), and also in case ii), except when  $d^2 - 4(r-1)(g-1) = 1$  and  $3d \equiv \pm 3 \pmod{2r-2}$  or  $d^2 - 4(r-1)(g-1) = 9$  or  $d \equiv \pm 3 \pmod{2r-2}$ , in which case  $S$  has to be a complete intersection of both quadrics and cubics.

## 4.2 An Important Theorem from Saint-Donat

The second problem is that we will study  $K3$  surfaces which are embedded into a projective space. This means that we must show existence of a suitable very ample divisor on  $S$ . The following lemma will be useful.

**Lemma 4.4.** ([SD74]) *Let  $L$  be a nef line bundle on a  $K3$  surface. Then*

- a)  $|L|$  is not base point free if and only if there exists curves  $E, \Gamma$  and an integer  $k \geq 2$  such that

$$L \sim kE + \Gamma, \quad E^2 = 0, \quad \Gamma^2 = -2, \quad \text{and } E \cdot \Gamma = 1.$$

*In this case, every member of  $|L|$  is of the form  $E_1 + \dots + E_k + \Gamma$ , where  $E_i \in |E|$  for all  $i$ . Equivalently,  $L$  is not base point free if and only if there is a divisor  $E$  satisfying  $E^2 = 0$  and  $E \cdot L = 1$ .*

- b)  $L$  is very ample if and only if  $L^2 \geq 4$  and

*i) there is no divisor  $E$  such that  $E^2 = 0$ ,  $E \cdot L = 1, 2$ ,*

- ii) there is no divisor  $E$  such that  $E^2 = 2$ ,  $L \sim 2E$ , and
- iii) there is no divisor  $E$  such that  $E^2 = -2$ ,  $E.L = 0$ ,

Note that b) ii) is immediate if  $L$  is a part of a basis of  $\text{Pic}(S)$ .

### 4.3 Numerical Conditions for Nonspecial Linebundles on $S$

In this section we give numerical conditions for when a line bundle on a  $K3$  surface is nonspecial. This will be used in Chapter 7, when we will study smooth points in the Hilbert scheme.

**Proposition 4.5.** (*[Knu02, Proposition 3.1]*) *Let  $l \geq 1$  be an integer. We can find  $S$  and  $C$  as in Theorem 4.3 such that  $h^1(C', \mathcal{O}_{C'}(l)) = 0$  for all  $C' \in |C|$  if and only if*

$$d \leq 2l(r-1) \text{ or } dl > (r-1)l^2 + g.$$

*Remark 4.6.* Note that this proposition only holds in the case where  $S$  has the Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ . When we study  $K3$  surfaces of Picard rank 3 in Chapter 7 the proposition above does not hold, but we will make some tricks so that it holds in some cases.

### 4.4 Results on the Three Invariants

Recall our discussion about constant Clifford index on page 31: M. Green and R. Lazarsfeld [GL87] showed if  $L$  is a base point free line bundle on a  $K3$  surface  $S$  then  $\text{Cliff}(C)$  is constant for all smooth irreducible  $C \in |L|$  and in addition they showed that if  $\text{Cliff}(C) < \lfloor \frac{g-1}{2} \rfloor$ , then  $M$  on  $S$  such that  $M_C := M \otimes \mathcal{O}_C$  computes the Clifford index of  $C$  for all smooth irreducible  $C \in |L|$ . Note also that since  $(L - M) \otimes \mathcal{O}_C \cong \omega_C \otimes M_C^{-1}$ , the result is symmetric in  $M$  and  $L - M$ .

**Lemma 4.7.** (*[Knu01, Lemma 8.3]*) *Let  $L$  be a base point free line bundle on a  $K3$  surface  $S$  with  $L^2 = 2g - 2 \geq 2$ , and let  $C$  be any smooth curve  $C \in |L|$ . If  $\text{Cliff}(C) < \lfloor \frac{g-1}{2} \rfloor$ , then there exists a smooth curve  $D$  on  $S$*

satisfying  $0 \leq D^2 \leq \text{Cliff}(C) + 2$ ,  $2D^2 \leq D.L$  (either of the two inequalities being an equality if and only if  $L \sim 2D$ ) and

$$\text{Cliff}(C) = \text{Cliff}(\mathcal{O}_S(D) \otimes \mathcal{O}_C) = D.L - D^2 - 2.$$

**Lemma 4.8.** *Let  $L$  be a base point free line bundle on a  $K3$  surface  $S$  with  $L^2 = 2g - 2 \geq 2$ , and let  $C$  be any smooth curve  $C \in |L|$ . Define the family of effective divisors on  $S$  by*

$$\mathcal{A} = \{D \in \text{Div}(S) \mid D^2 \geq 0, 0 < C.D \leq g - 1\}.$$

*If  $\text{Cliff}(C) < \lfloor \frac{g-1}{2} \rfloor$ , then  $\text{Cliff}(C) = \min\{D.C - D^2 - 2 \mid D \in \mathcal{A}\}$  for all  $C \in |L|$ . In particular, if  $C$  is not in the Donagi-Morrison example or the generalized ELMS examples, then  $\text{gon}(C) = \min\{D.C - D^2 \mid D \in \mathcal{A}\}$ .*

*Proof.* First, we must show that the smooth curve in Lemma 4.7 is an element of the set  $\mathcal{A}$ .  $D^2 \geq 0$  is clearly satisfied. We are left to show that  $D.C \leq g - 1$ . By symmetry, we can assume  $M.L \leq (L - M).L$  or equivalently  $2M.L \leq L^2$ , where  $M$  and  $L$  are the line bundles in [GL87] or see our discussion just before Lemma 4.7.  $C$  and  $D$  are elements of  $|L|$  and  $|M|$ , respectively. Therefore,  $2D.C \leq C^2$  which is the same as  $D.C \leq g - 1$ , so  $D$  is in  $\mathcal{A}$ . Given an effective divisor  $D \in \mathcal{A}$ , then  $D$  must satisfy the three conditions  $D^2 \geq 0$ ,  $(C - D)^2 \geq 0$  and  $C.D \leq g - 1$ . We see that  $D^2 \geq 0$ ,  $C.D \leq g - 1$  implies  $(C - D)^2 \geq 0$ , i.e.

$$\begin{aligned} (C - D)^2 &= C^2 - 2C.D + D^2 \\ &\geq 2g - 2 - 2(g - 1) = 0. \end{aligned}$$

So the condition  $(C - D)^2 \geq 0$  is not necessary. The Riemann-Roch theorem gives that  $h^0(\mathcal{O}_S(D)) \geq 2$  and  $h^0(\mathcal{O}_S(C - D)) \geq 2$ . Now Lemma 4.7 says that  $\text{Cliff}(C) = D.C - D^2 - 2$  for some smooth curve  $D \in \mathcal{A}$ , hence the inequality

$$\text{Cliff}(C) \geq \min\{D.C - D^2 - 2 \mid h^0(\mathcal{O}_C(D)) \geq 2, h^1(\mathcal{O}_C(D)) \geq 2\}.$$

We want to show the other inequality. The standard exact sequence

$$0 \longrightarrow \mathcal{O}_S(D - C) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D) \otimes \mathcal{O}_C \longrightarrow 0. \quad (4.4.1)$$

Since  $C \in |L|$  and  $D \in |M|$ ,  $\mathcal{O}_S(L - M) \cong \mathcal{O}_S(C - D)$ . From the Riemann-Roch theorem for curves, we have

$$h^0(\mathcal{O}_S(D) \otimes \mathcal{O}_C) = h^1(\mathcal{O}_S(D) \otimes \mathcal{O}_C) + D.C + 1 - g, \quad (4.4.2)$$



and the long exact sequence of cohomology of (4.4.1) combined with the fact that  $h^1(\mathcal{O}_S(D - C)) = h^1(\mathcal{O}_S(L - M)) = 0$ , by [Mar89, (2,3)], gives that

$$\begin{aligned} h^0(\mathcal{O}_S(D) \otimes \mathcal{O}_C) &= h^0(\mathcal{O}_S(D)) \\ &= \frac{1}{2}D^2 + 2 + h^1(\mathcal{O}_S(D)) - h^2(\mathcal{O}_C(D)) \\ &\geq \frac{1}{2}D^2 + 2, \end{aligned}$$

by Serre duality and that  $D$  is effective. Hence

$$\begin{aligned} \text{Cliff}(C) &= \text{Cliff}(\mathcal{O}_S(D) \otimes \mathcal{O}_C) \\ &= \deg(\mathcal{O}_S(D) \otimes \mathcal{O}_C) - 2[h^0(\mathcal{O}_S(D) \otimes \mathcal{O}_C) - 1] \\ &\leq C.D - 2\left(\frac{1}{2}D^2 + 1\right) \\ &= C.D - D^2 - 2. \end{aligned}$$

Since this is valid for all smooth curves  $C \in |L|$ , we get that  $\text{Cliff}(C) \leq \min\{D.L - D^2 - 2 | h^0(\mathcal{O}_C(D)) \geq 2, h^1(\mathcal{O}_C(D)) \geq 2\}$ , which means

$$\text{Cliff}(C) = \min\{D.C - D^2 - 2 | D \in \mathcal{A}\},$$

for any smooth curve  $C \in |L|$ . If  $C$  is not the Donagi-Morrison example or the generalized ELMS examples, the gonality is constant within  $|L|$  and no  $C \in |L|$  are exceptional, hence

$$\text{gon}(C) = \text{Cliff}(C) + 2 = \min\{D.C - D^2 | D \in \mathcal{A}\}.$$

□

In the next Corollary we wanted to express all exceptional curves satisfying Theorem 4.3 in terms of  $d$ ,  $g$  and  $r$ . Due to the time limit of this thesis, we was not able to classify all exceptional curves. But it can be useful if we are working with  $K3$ s and want to eliminate the Donagi-Morrison example and the generalized ELMS examples.

**Corollary 4.9.** *If  $C$  is an exceptional curve of degree  $d$  and genus  $g$  satisfying Theorem 4.3, then*

$$d^2 = (g - 1)(4r - 3),$$

*with  $d$  odd and  $g$  even.*

*Proof.* Suppose that  $C$  is an exceptional curve on a  $K3$  surface, then  $C$  is either the smooth plane sextic in the Donagi-Morrison example 3.14 or as in the generalized ELMS example 3.13, by [Knu09, Theorem 1.2]. If  $C$  belongs to the Donagi-Morrison example, it must be on the form  $C \sim 3R$ , where  $R \in |\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$ . But this cannot happen, because  $C$  is a basiselement of  $\text{Pic}(S)$ .

If  $C$  satisfies the generalized ELMS examples, then  $|C|$  is a base point free linear system and all the smooth curves  $|C|$  are exceptional. The generalized ELMS example says that  $C$  must be linear equivalent to a divisor on the form  $2D + \Gamma$ , where  $\Gamma$  is a rational curve satisfying  $D \cdot \Gamma = 1$ , and  $D^2 \geq 0$ . This implies that  $C \cdot \Gamma = 2D \cdot \Gamma + \Gamma^2 = 0$ , hence

$$2g - 2 = C^2 = C \cdot (2D + \Gamma) = 2C \cdot D,$$

so  $C \cdot D = g - 1$ . Moreover,  $\Gamma \in \text{Pic}(S)$ , we write  $\Gamma \sim xH + yC$ .  $\Gamma^2 = \Gamma \cdot (xH + yC) = x\Gamma \cdot H = -2$ .  $H$  is very ample, so  $\Gamma \cdot H > 0$ . This gives two possibilities

- i)  $x = -1$  and  $\Gamma \cdot H = 2$ ,
- ii)  $x = -2$  and  $\Gamma \cdot H = 1$ .

Consider the case i):

$\Gamma \sim -H + yC$ . To find  $y$ , we do the following:

$$2 = \Gamma \cdot H = (-H + yC) \cdot H = -2n + yd \text{ implies } y = \frac{2 + 2n}{d}.$$

Hence  $\Gamma \sim -H + \frac{2n+2}{d}C$ . Using that  $\Gamma \cdot C = 0$ , we obtain that

$$\begin{aligned} \Gamma \cdot C &= (-H + yC) \cdot C \\ &= -d + (2g - 2)y \\ &= 0, \end{aligned}$$

which gives  $y = \frac{d}{2g-2}$ . Equating the two expressions for  $y$ , we obtain

$$d^2 = 4(g - 1)(n + 1).$$

Consider case ii):

$\Gamma \sim -2H + yC$ , to find  $y$ , we do the same as above:

$$1 = \Gamma \cdot H = (-2H + yC) \cdot H = -4n + yd \text{ implies } y = \frac{1 + 4n}{d}.$$

Hence  $\Gamma \sim -2H + \frac{1+4n}{d}C$ . On the other hand  $\Gamma.C = 0$  implies that  $y = \frac{d}{g-1}$ . Equating and we obtain

$$d^2 = (g-1)(4n+1).$$

We will now show that we can eliminate the case where  $\Gamma \sim -H + \frac{2+2n}{d}C$ .

We write  $D \sim x'H + y'C$  for integers  $x'$  and  $y'$ .  $D.\Gamma = (x'H + y'C).\Gamma = x'H.\Gamma = 1$ , which implies that  $x' = 1$  and  $\Gamma.H = 1$  is the only possibility. Hence, the only divisor that can satisfy the generalized ELMS examples is  $\Gamma \sim -2H + \frac{1+4n}{d}C$ . To find the value of  $y'$  we do the following: Let  $D \sim H + y'C$ , then  $C - \Gamma \sim 2D \sim 2H + 2y'C$ . Then on the one hand

$$C.2D = C.(C - \Gamma) = 2g - 2.$$

On the other hand

$$\begin{aligned} C.2D &= C.(2H + 2y'C) \\ &= 2d + 4(g-1)y'. \end{aligned}$$

This implies that  $y' = \frac{1}{2} \left(1 - \frac{d}{g-1}\right)$ . So  $D \sim H + \frac{1}{2} \left(1 - \frac{d}{g-1}\right) C$ . One can calculate very easily that  $D.H = \frac{d+1}{2}$ , using that  $d^2 = (g-1)(4n+1)$ . This means that  $d$  is odd and  $g$  even.  $\square$

*Remark 4.10.* Notice that the exceptional curves occur, if any, only in situation iv) of Theorem 4.3.

## 4.5 Some Results From Kley

The last 4 results in this chapter can be found in the paper [Kle00]. We have filled in some details in the proof the most important result of this section, namely Lemma 4.14. It will be important for us in Chapter 7, when we study smooth points in the Hilbert scheme. Let us look at the case  $S \hookrightarrow \mathbb{P}^r$ , where  $S$  is a smooth  $K3$  surface,  $C_0$  a smooth connected curve of genus  $g$  on  $S$ , and  $\mathcal{O}_S(C_0)$  is the divisorial sheaf of  $C_0$ . Denote  $|\mathcal{O}_S(C_0)|$  as the projective space bundle  $\mathbb{P}(\Gamma(S, \mathcal{O}_S(C_0)))$ .

**Lemma 4.11.** ([Kle00, Lemma 1.9]) *For all  $C \in |\mathcal{O}_S(C_0)|$  and  $m > 0$ :*

- i)  $h^0(\mathcal{O}_C) = 1$  and  $h^1(\mathcal{O}_C) = g$ ,
- ii)  $h^0(\mathcal{N}_{C/S}) = g$  and  $h^1(\mathcal{N}_{C/S}) = 1$ , and
- iii) If  $\mathcal{O}_{C_0}(m)$  is nonspecial, then  $\mathcal{O}_C(m)$  is nonspecial.

**Proposition 4.12.** ([Kle00, Proposition 1.7]) Suppose  $S$  is projective:  $S \hookrightarrow \mathbb{P}^r$ . Then for all  $k$

$$H^{k-2}(S, \mathcal{N}_{S/\mathbb{P}^r}) \xrightarrow{\delta} H^{k-1}(S, \mathcal{T}_S) \xrightarrow{c} H^k(S, \mathcal{O}_S)$$

is a complex, exact if  $H^{k-1}(S, \mathcal{O}_S(1)) = 0$ . The map  $\delta$  is the connecting homomorphism arising from

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_S \longrightarrow \mathcal{N}_{S/\mathbb{P}^r}$$

and  $c$  is the Yoneda Pairing<sup>1</sup>  $\times c_1(\mathcal{O}_S(1))$ .

Let  $\pi : \mathbf{P} \longrightarrow X$  be the associated projective space bundle to a line bundle  $\mathcal{O}_S(C_0)$  on  $X$ .

**Proposition 4.13.** ([Kle00, Proposition 1.8]) If  $X$  is a smooth projective variety over  $\mathbb{C}$  and  $D$  an effective divisor on  $X$ , then the square

$$\begin{array}{ccc} H^{k-1}(X, \mathcal{T}_X) & \xrightarrow{\times 2\pi i c_1(\mathcal{L})} & H^k(X, \mathcal{O}_X) \\ \downarrow H^{k-1}(\mu) & & \parallel \\ H^{k-1}(X, \mathcal{O}_S(C_0) \otimes \mathcal{O}_D)^\delta & \longrightarrow & H^k(X, \mathcal{O}_X) \end{array}$$

commutes for all  $k$ .

**Lemma 4.14.** ([Kle00, Lemma 1.10]). Let  $S$  be a K3 surface. Suppose that  $\mathcal{O}_S(1)$  and  $\mathcal{O}_S(C_0)$  are independent in  $\text{Pic}(S)$ . Then for all  $C \in |\mathcal{O}_S(C_0)|$ , the composition

$$\phi : H^0(S, \mathcal{N}_{S/\mathbb{P}^r}) \longrightarrow H^0(C, \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{N}_{C/S})$$

---

<sup>1</sup>Let  $\mathcal{F}$  and  $\mathcal{G}$  be  $\mathcal{O}_X$ -modules on a scheme  $X$ , then there is a  $\delta$ -functorial pairing

$$H^{r-i}(X, \mathcal{F}) \times \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{G}) \longrightarrow H^r(X, \mathcal{G}),$$

called the *Yoneda Pairing*.

of the restriction with the connecting homomorphism arising from the exact sequence

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbb{P}^r} \longrightarrow \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C \longrightarrow 0, \quad (4.5.1)$$

is surjective. Furthermore, the kernel  $\ker \phi$  is independent of  $C$ .

*Proof.* Combining the exact sequence (2.4.4) with (4.5.1), we obtain a commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}_S & \longrightarrow & \mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_S & \longrightarrow & \mathcal{N}_{S/\mathbb{P}^r} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{N}_{C/S} & \longrightarrow & \mathcal{N}_{C/\mathbb{P}^r} & \longrightarrow & \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C \longrightarrow 0 \end{array}$$

of  $\mathcal{O}_S$ -modules. This gives rise to a commutative diagram

$$\begin{array}{ccccc} H^0(S, \mathcal{N}_{S/\mathbb{P}^r}) & \xrightarrow{\delta'} & H^1(S, \mathcal{T}_S) & \xrightarrow{\times c_1(\mathcal{L})} & H^2(S, \mathcal{O}_S) \\ \downarrow & & \downarrow & & \parallel \\ H^0(C, \mathcal{N}_{C/\mathbb{P}^r} \otimes \mathcal{O}_C) & \longrightarrow & H^1(C, \mathcal{N}_{C/S}) & \xrightarrow{\delta} & H^2(S, \mathcal{O}_S) \end{array}$$

We see that the  $\delta'$  map is the connecting homomorphism of the sequence (2.4.4) and  $\delta$  is the connecting homomorphism of (4.5.1). The rightmost square comes from Proposition 4.13. Now  $\delta$  is an isomorphism, since

$$h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) = 1,$$

and  $h^1(S, \mathcal{N}_{C/S}) = 1$  by Lemma 4.11 ii). The top row is independent of  $C$ , so is  $\ker \phi$ . To prove surjectivity, it is enough to show that the composition  $(\times c_1(\mathcal{O}_S(C_0))) \circ \delta'$  is surjective. We do this by showing that  $\text{im}(\delta') \subsetneq \ker(\times c_1(\mathcal{O}_S(C_0)))$ . On  $K3$  surfaces  $\omega_S \cong \mathcal{O}_S$  and note that

$$\text{Ext}_{\mathcal{O}_S}^1(\mathcal{T}_S, \mathcal{O}_S) \cong H^1(S, \mathcal{O}_S \otimes \mathcal{T}_S^\vee) \cong H^1(S, \Omega_S^1).$$

Serre duality on  $S$  states that the Yoneda pairing

$$H^1(S, \mathcal{T}_S) \times H^1(S, \Omega_S^1) \longrightarrow H^2(S, \mathcal{O}_S)$$

is non-degenerate. The maps  $\times c_1(\mathcal{O}_S(C_0))$  and  $\times c_1(\mathcal{O}_S(1))$  are surjective since  $\mathcal{O}_S(C_0)$  and  $\mathcal{O}_S(1)$  are independent in  $\text{Pic}(S)$ , hence

$$\ker(\times c_1(\mathcal{O}_S(1))) \neq \ker(\times c_1(\mathcal{O}_S(C_0))).$$

The hyperplane section  $\mathcal{O}_S(1)$  is very ample,  $h^1(S, \mathcal{O}_S(1)) = 0$  by Kodaira Vanishing theorem (Theorem 2.14). Hence, Proposition 4.12 gives  $\text{im}(\delta') = \ker(\times c_1(\mathcal{O}_S(1)))$ .  $\square$



# Chapter 5

## The Gonality of Curves in $\mathbb{P}^r$

### 5.1 Brill-Noether Theory and the Moduli Space

$\mathcal{M}_g$

The most fundamental question of *Brill-Noether theory* is:

*For which values of  $r$  and  $d$  does a  
general curve of genus  $g$  possess a  $\mathfrak{g}_d^r$ ?*

The answer to this question is the famous Brill-Noether Theorem<sup>1</sup> which asserts that when the *Brill-Noether number*

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is negative, the general curve of genus  $g$  has no  $\mathfrak{g}_d^r$ 's. If we think of the  $\mathfrak{g}_d^r$  as being given by an  $(r + 1)$ -dimensional subspace  $W$  of the space of sections  $H^0(C, L)$  of some degree  $d$  line bundle  $L$  on  $C$ , then the nonnegativity of the Brill-Noether number says that the dimension of the codomain of the cup-product map (or Petri map)

$$\mu_{0,W} : W \otimes H^0(C, \omega_C \otimes L^\vee) \longrightarrow H^0(C, \omega_C)$$

is greater than or equal to its domain. This leads us to the following statement: If a smooth curve  $C$  possesses a  $\mathfrak{g}_d^r$  with negative Brill-Noether number, then the curve is special in the sense of moduli.

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<sup>1</sup>See Theorem 2.3, Chapter V in [ACH85].

For genera  $g \geq 3$  we consider the stratification of the moduli space  $\mathcal{M}_g$  of smooth curves of the genus  $g$  given by gonality:

$$\mathcal{M}_{g,2}^1 \subseteq \mathcal{M}_{g,3}^1 \subseteq \dots \mathcal{M}_{g,k}^1 \subseteq \dots \subseteq \mathcal{M}_g,$$

where

$$\mathcal{M}_{g,k}^1 := \{|C| \in \mathcal{M}_g \mid C \text{ has a } \mathfrak{g}_k^1\}.$$

It is well-known that the  $k$ -gonal locus  $\mathcal{M}_{g,k}^1$  is an irreducible variety of dimension  $2g+2k-5$  when  $k \leq \frac{g+2}{2}$  and when  $k \geq \frac{g+3}{2}$  one has that  $\mathcal{M}_{g,k}^1 = \mathcal{M}$  (see [AC81]). As we have seen earlier, the number  $\lfloor \frac{g+3}{2} \rfloor$  is the gonality of a generic curve of genus  $g$ .

$\mathcal{M}_{g,d}^1$  can easily be generalized. For positive integers  $g, d$  and  $r$  we define  $\mathcal{M}_{g,d}^r$  in the obvious way

$$\mathcal{M}_{g,d}^r = \{|C| \in \mathcal{M}_g \mid C \text{ carries a } g_d^r\},$$

which is known as the *Brill-Noether locus*. When

$$\rho(g, r, d) = g - (r+1)(g-d+r)$$

is negative, the Brill-Noether locus  $\mathcal{M}_{g,d}^r$  is a proper subvariety of  $\mathcal{M}_g$ . We want to know the gonality of a point  $[C] \in \mathcal{M}_{g,d}^r$ , which is equivalent to knowing the gonality of a general smooth curve  $C \subseteq \mathbb{P}^r$  of genus  $g$  and degree  $d$ . The geometry of the loci  $\mathcal{M}_{g,d}^r$  is not easy to work with, because there are many components and some of them are nonreduced and/or not of expected dimension. In this section we will calculate the gonality of components of  $\mathcal{M}_{g,d}^r$  which are generically smooth.

## 5.2 The Expected Gonality

From Lemma 4.8 we saw that the gonality of a smooth curve  $C \subseteq \mathbb{P}^r$ , where  $C$  is a  $K3$  section, can be calculated using the formula

$$\text{gon}(C) = \min \{D.C - D^2 \mid D^2 \geq 0, 0 \leq D.C \leq g-1\},$$

for  $D \in \text{Pic}(S)$ , unless we are in the Donagi-Morrison example or in the generalized ELMS examples. If  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ , note that  $D$  can be



written as  $mH + nC$ , therefore the gonality can be calculated numerically, by finding the minimum

$$\min\{f(m, n) := -(2r - 2)m^2 - m(d - 2nd) + (n - n^2)(2g - 2)\}, \quad (5.2.1)$$

bounded by  $D^2 \geq 0$  and  $C.D \leq g - 1$ . Quoting G. Martens in [Mar06, p. 125]:

*In general, one expects that  $|H|_C|$  computes  $\text{Cliff}(C)$  for  $C \subset S$  in Lemma 3.1; it seems, however, difficult to make this precise.*

This means that he expects that it is the hyperplane section which gives the gonality in general. G. Farkas shows [Far01, Theorem 3] that it is indeed so, under the assumptions that the  $K3$  surface does not contain  $-2$ -curves and curves with genus 1. That is, saying that 0 and  $-1$  cannot be represented by the quadratic form

$$\frac{D^2}{2} = (r - 1)m^2 + mnd + n^2(g - 1), \text{ for integers } m, n \in \mathbb{Z}.$$

We see that  $\frac{D^2}{2} \neq -1$  when for instance  $r$  is odd,  $d$  is even and  $g$  is odd. A necessary condition for  $D^2$  to represent 0 is that  $\sqrt{d^2 - 4(r - 1)(g - 1)}$  is an integer, which can easily be seen by solving the quadratic form above with respect to  $m$ . Farkas call the number

$$\min \left\{ d - 2r + 2, \left\lfloor \frac{g + 3}{2} \right\rfloor \right\} \quad (5.2.2)$$

the *expected gonality*, with good reasons, of a smooth nondegenerate curve  $C \subseteq \mathbb{P}^r$  of degree  $d$  and genus  $g$ . Notice that  $D \sim H$  and  $D \sim C - H$ , gives the number  $d - 2r + 2$ , because if we insert  $m = 1$ ,  $n = 0$  and  $m = -1$ ,  $n = 1$  respectively into formula for  $f(m, n)$  (5.2.1) obtain  $d - 2r + 2$ . The last possibility can be dismissed, because it is not compatible with the assumptions in [Far01, Theorem 3].

In the next section will show, in Example 5.2 below, that the formula for the expected gonality does not hold in general. We will generalize Farkas' result by simplifying an inequality and allow  $-2$ -curves and genus 1 curves.

### 5.3 The Gonality of $K3$ Sections in $\mathbb{P}^r$

This section is devoted to a generalization of Theorem 3 in [Far01]. We will calculate the gonality of curves  $C \subseteq \mathbb{P}^r$  of degree  $d$  and genus  $g$  when we look the curves as sections of  $K3$  surfaces.

**Theorem 5.1.** *Let  $r \geq 3$ ,  $d \geq r^2 + r$  and  $g \geq 0$  be integers such that  $\rho(g, r, d) < 0$  and with  $d^2 > 4(r-1)g$ . Then there exists a smooth curve  $C \subseteq \mathbb{P}^r$  of degree  $d$  and genus  $g$  with  $(d, g) \neq (2r-1, r)$  such that*

$$\text{gon}(C) = \min\{N_r, d - 2r - 2, \lfloor (g+3)/2 \rfloor\},$$

where

$$N_r = \min\{f(m, n) \mid n \in \mathbb{Z}, 0 < |n| < r-1 \\ \text{and } md + (2n-1)(g-1) \leq 0\},$$

and

$$m = \left\lceil \frac{[2|n|\sqrt{r-1}] + \epsilon - nd}{2r-2} \right\rceil,$$

$\epsilon$  is defined as

$$\epsilon = \begin{cases} 1 & , \text{ if } \sqrt{r-1} \in \mathbb{Z} \\ 0 & , \text{ if } \sqrt{r-1} \notin \mathbb{Z} \end{cases}$$

and  $f$  is given by the formula

$$f(m, n) = -(2r-2)m^2 + m(d-2nd) + (n-n^2)(2g-2). \quad (5.3.1)$$

*Proof.* By Theorem 4.3 there exists a smooth  $K3$  surface  $S \subseteq \mathbb{P}^r$  with  $\deg(S) = 2r-2$  and  $C \subseteq S$  a smooth curve of degree  $d$  and genus  $g$  with  $d^2 > 4(r-1)g$ , unless  $(d, g) = (2r-1, r)$ , such that

$$\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C,$$

where  $H$  is a hyperplane section. The Brill-Noether number  $\rho(g, r, d) < 0$  together with the assumption  $d \geq r^2 + r$  gives

$$\begin{aligned} \rho(g, r, d) &= g - (r+1)(g-d+r) \\ &= -rg + (r+1)d - (r^2+r) \\ &\geq r(d-g), \end{aligned}$$

hence the boundary  $d \leq g - 1$ . We will use this boundary only once in the proof. Now we want to show that if  $\text{gon}(C) < \lfloor (g + 3)/2 \rfloor$ , then  $\text{gon}(C) = \min\{N_r, d - 2r + 2\}$ . Define

$$\alpha := \min\{D.C - D^2 \mid D \in \mathcal{A}\},$$

Note that the line bundle  $\mathcal{O}_C(1)$  contributes to the Clifford index, because the assumption  $d \leq g - 1$  together with the Riemann-Roch theorem gives  $h^0(\mathcal{O}_C(1)) \geq 2$  and  $h^1(\mathcal{O}_C(1)) \geq 2$ . Now we want to prove that

$$\alpha \geq d - 2r - 2 \text{ or } \alpha \geq N_r. \quad (5.3.2)$$

We will derive three boundary conditions, which we will use to show (5.3.2).

Take  $D \in \mathcal{A}$  such that  $D \sim mH + nC$ , with  $m, n \in \mathbb{Z}$ . The conditions  $D^2 \geq 0$ ,  $D.C \leq g - 1$ , can be expressed as  $(r - 1)m^2 + mnd + n^2(g - 1) \geq 0$  and  $md + (2n - 1)(g - 1) \leq 0$ , respectively.

Moreover, we know  $H$  is very ample and  $D$  is effective, thus  $D.H > 0$ . We know that  $D^2 \geq 0$ , so if  $D^2 = 0$  we have that  $D.H$  cannot be equal to 1 and 2 by Lemma 4.4 b) (i), so  $D.H > 2$  in this case. If  $D^2 > 0$ , which is an even number, thus  $(D.H)^2 \geq 4$  by the Hodge Index theorem. Suppose we have equality,  $(D.H)^2 = H^2.D^2 = 4$ , then Proposition 3.4 tells us that  $(D.H)H \sim H^2D$ . This gives that  $H \sim 2D$ , which is impossible, by Lemma 4.4 b) (ii)<sup>2</sup>. We conclude that  $D.H > 2$ . On the other hand we have that  $H.(C - D) < d - 2$ , which gives the inequality  $2 < D.H < d - 2$ , using a symmetry argument. We summarize the boundary conditions we will use to show (5.3.2):

- i)  $(r - 1)m^2 + mnd + n^2(g - 1) \geq 0$ ,
- ii)  $2 < (2r - 2)m + nd < d - 2$
- iii)  $md + (2n - 1)(g - 1) \leq 0$ .

The gonality of a canonical curve  $C$  lying on a  $K3$  surface can be calculated by using the equation we get from Lemma 4.8,

$$f(m, n) = D.C - D^2 = -(2r - 2)m^2 + m(d - 2nd) + (n - n^2)(2g - 2).$$

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<sup>2</sup>We could also see this directly, because  $H$  is a generator of  $\text{Pic}(S)$ .

We must show that for all  $D \in \mathcal{A}$ , we have the following inequality

$$\begin{aligned} f(m, n) &= D.C - D^2 = -(2r - 2)m^2 + m(d - 2nd) + (n - n^2)(2g - 2) \\ &\geq d - 2r + 2 \text{ or } N_r. \end{aligned} \quad (5.3.3)$$

From i) we solve  $h(m) = (r - 1)m^2 + mnd + n^2(g - 1) = 0$  to obtain the factorization

$$h(m) = (r - 1)(m + an)(m + bn), \quad (5.3.4)$$

where

$$a = \frac{d + \sqrt{d^2 - 4(r - 1)(g - 1)}}{2r - 2} \text{ and } b = \frac{d - \sqrt{d^2 - 4(r - 1)(g - 1)}}{2r - 2}.$$

We solve the problem numerically. To analyze the behavior of  $f(m, n)$ , it is natural to study the partial derivatives of  $f$ . Consider  $f(m, n)$  as a real valued function of two variables. The partial derivative with respect to  $m$  and  $n$  gives

$$\frac{\partial f(m, n)}{\partial m} = -4(r - 1)m + d(1 - 2n) \quad (5.3.5)$$

and

$$\frac{\partial f(m, n)}{\partial n} = -2dm + (1 - 2n)(2g - 2). \quad (5.3.6)$$

Now we can start to show the inequality (5.3.13). We will consider the cases where  $n > 0$  and  $n < 0$ , separately.

**Case 1:**  $n < 0$ . From (5.3.4) it follows that either  $m \leq -bn$  or  $m \geq -an$ .

**If:**  $m \leq -bn$ . From ii), we obtain

$$2 < n(d - (2r - 2)b) < 0,$$

because  $n < 0$  and

$$d - (2r - 2)b = \sqrt{d^2 - 4(r - 1)(g - 1)} > 0,$$

thus a contradiction.

**If:**  $m \geq -an$ . From iii), we have

$$m \leq \frac{(g - 1)(1 - 2n)}{d}.$$

If  $-an > (g-1)(1-2n)/d$  we are done, because there are no  $m, n \in \mathbb{Z}$  satisfying i) ,ii) and iii), while in the other case, i.e., where

$$-an \leq m \leq \frac{(g-1)(1-2n)}{d}$$

one has the inequality

$$f(m, n) \geq f(-an, n) = \frac{d^2 - 4(r-1)(g-1) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(-n).$$

If  $\sqrt{d^2 - 4(r-1)(g-1)} \geq 2r-2$ , we get that

$$\begin{aligned} f(m, n) \geq f(-an, n) &= \frac{d^2 - 4(r-1)(g-1) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(-n) \\ &\geq \frac{(2r-2)^2 + (2r-2)d}{2r-2}(-n) \\ &= (2r-2+d)(-n) \\ &\geq d-2r+2. \end{aligned}$$

From now on we consider the situation where

$$\sqrt{d^2 - 4(r-1)(g-1)} < 2r-2. \quad (5.3.7)$$

Now we must look for what values of  $n$  we get  $f(m, n) \geq d-2r+2$  or  $N_r$ . We see that if  $n \leq -(r-1)$  then

$$\begin{aligned} f(m, n) \geq f(-an, n) &= \frac{d^2 - 4(r-1)(g-1) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(-n) \\ &\geq \frac{d^2 - 4(r-1)(g-1) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(r-1) \\ &> \frac{4(r-1) + 2\sqrt{(r-1)}d}{2} \\ &= 2(r-1) + \sqrt{(r-1)}d > d-2r+2. \end{aligned}$$

Here we have only used that  $d^2 > 4(r-1)g$ . We obtain from the inequality  $m \geq -an$  that

$$m \geq \begin{cases} \frac{(-n)d + \left\lceil \frac{2(-n)\sqrt{r-1}}{2r-2} \right\rceil}{2r-2} & , \text{ if } \sqrt{r-1} \notin \mathbb{Z} \\ \frac{(-n)d + \left\lceil \frac{2(-n)\sqrt{r-1}}{2r-2} \right\rceil + 1}{2r-2} & , \text{ if } \sqrt{r-1} \in \mathbb{Z}. \end{cases}$$

To find the minimum, we must find how the function behave for different values of  $m$  and  $n$ . Using iii), we obtain

$$m \leq -\frac{(2n-1)(g-1)}{d}. \quad (5.3.8)$$

Insert (5.3.8) into the partial derivative of  $f$  with respect  $m$  (5.3.5) we get

$$\begin{aligned} \frac{\partial f}{\partial m} &\geq 4(r-1) \left[ -\frac{(2n-1)(g-1)}{d} \right] + d(1-2n) \\ &= \frac{-4(r-1)(g-1)(2n-1) + d^2(1-2n)}{d} \\ &= \frac{(d^2 + 4(r-1)(g-1))(1-2n)}{d} \\ &> 0, \end{aligned}$$

since  $n < 0$ .

For fixed  $m$  we use (5.3.8) again,

$$\begin{aligned} \frac{\partial f}{\partial n} &\geq 2(2n-1)(g-1) + (1-2n)(2g-2) \\ &= 0, \end{aligned}$$

with equality if and only if  $D.C = g-1$ . This means that the minimum in the case  $n < 0$  will be

$$\min \left\{ f \left( \left\lceil \frac{(-n)d + 2(-n)\sqrt{r-1} + \epsilon}{2r-2} \right\rceil, n \right) \mid -(r-1) < n < 0 \right\},$$

where  $\epsilon = 0$  if  $\sqrt{r-1}$  is not an integer and  $\epsilon = 1$  otherwise.

**Case 2:**  $n > 0$ . This situation we can treat in a similar manner. From (5.3.4) we get that either  $m \leq -an$  or  $m \geq -bn$ .

**If:**  $m \leq -an$ . From ii) we get that

$$2 < n(d - (2r-2)a) < 0,$$

because  $n > 0$  and

$$d - (2r-2)a = -\sqrt{d^2 - 4(r-1)(g-1)} < 0,$$

so we have a contradiction.

**If:**  $m \geq -bn$ . Let us first examine the behavior of  $f(m, n)$  when  $n > 0$ . We must look at the partial derivatives (5.3.5) and (5.3.6). For the partial derivative of  $f$  with respect  $m$  on the interval from ii) and iii)

$$\max \left\{ \frac{2 - nd}{2r - 2}, -bn \right\} < m < \min \left\{ -\frac{(2n - 1)(g - 1)}{d}, \frac{(1 - n)d - 2}{2r - 2} \right\}, \quad (5.3.9)$$

the function  $f(m, n)$  is concave down. The maximum of this function when  $n$  is fixed is at point where

$$m = \frac{d(1 - 2n)}{4(r - 1)}. \quad (5.3.10)$$

This means that the minimum value of  $f(m, n)$  are at the end points of the interval (5.3.9). We can easily calculate the number

$$\min \left\{ -\frac{(2n - 1)(g - 1)}{d}, \frac{(1 - n)d - 2}{2r - 2} \right\}. \quad (5.3.11)$$

Both points are greater than  $m = \frac{d(1 - 2n)}{4(r - 1)}$ , i.e

$$\frac{d(1 - 2n)}{4(r - 1)} < -\frac{(2n - 1)(g - 1)}{d} \Rightarrow d^2 - 4(r - 1)(g - 1) > 0,$$

which is of course true, by assumption and

$$\frac{d(1 - 2n)}{4(r - 1)} < \frac{(1 - n)d - 2}{2r - 2} \Rightarrow 4 < d,$$

which is also true by assumption. Now we just compare these to endpoints, by using the fact that  $f$  is decreasing.

$$\begin{aligned} f \left( -\frac{(2n - 1)(g - 1)}{d}, n \right) &> f \left( \frac{(1 - n)d - 2}{2r - 2}, n \right) \\ \Rightarrow -\frac{(2n - 1)(g - 1)}{d} &< \frac{(1 - n)d - 2}{2r - 2}. \end{aligned}$$

We calculate<sup>3</sup>:

$$\begin{aligned}
& f\left(-\frac{(2n-1)(g-1)}{d}, n\right) - f\left(\frac{(1-n)d-2}{2r-2}, n\right) \\
&= \frac{g-1}{2} \left[ (2n-1)^2 \frac{d^2 - 4(r-1)(g-1)}{d^2} + 1 \right] \\
&\quad - \left[ (n^2 - n) \left( \frac{d^2 - 4(r-1)(g-1)}{2r-2} \right) + \frac{2(1-n)d}{2r-2} - \frac{4}{2r-2} \right] \\
&\geq \frac{1}{2} \left[ (4(n^2 - n) + 1) \frac{d^2 - 4(r-1)(g-1)}{d} + d \right] \\
&\quad - \left[ (n^2 - n) \left( \frac{d^2 - 4(r-1)(g-1)}{2r-2} \right) + \frac{2(1-n)d}{2r-2} - \frac{4}{2r-2} \right] \\
&= (n^2 - n) \left[ \frac{d^2 - 4(r-1)(g-1)}{2r-2} \right] + \frac{d^2 - 4(r-1)(g-1)}{2d} \\
&\quad + \frac{d}{2} - \frac{2}{2r-2} + \frac{2nd}{2r-2} + \frac{4}{2r-2} > 0.
\end{aligned}$$

Notice that we used that  $d \leq g-1$  and that  $f$  is decreasing. We use that for any  $D \in \mathcal{A}$  with  $D \sim mH + nC$ ,

$$f(m, n) \geq \min \left\{ f\left(\left\lfloor -\frac{(g-1)(2n-1)}{d} \right\rfloor, n\right), \max \left\{ f(\lceil -bn \rceil, n), f\left(\left\lceil \frac{2-nd}{2r-2} \right\rceil, n\right) \right\} \right\}. \quad (5.3.12)$$

Using that  $d \leq g-1$  and  $d^2 > 4(r-1)g$ , we obtain

$$\begin{aligned}
f\left(\left\lfloor -\frac{(g-1)(2n-1)}{d} \right\rfloor, n\right) &\geq f\left(-\frac{(g-1)(2n-1)}{d}, n\right) \\
&= \frac{g-1}{2} \left[ (2n-1)^2 \frac{d^2 - 4(r-1)(g-1)}{d^2} + 1 \right] \\
&\geq d - 2r + 2.
\end{aligned}$$

Note that we have equality if and only if  $n = 1$ ,  $m = -1$  and  $d = g-1$ . Therefore this element can be dismissed from (5.3.12). Now we want to show that if  $n \geq r-1$  then  $f(m, n) \geq d - 2r + 2$ . In other words to show that the set

$$\mathcal{C} := \left\{ f(\lceil -bn \rceil, n), f\left(\left\lceil \frac{2-nd}{2r-2} \right\rceil, n\right) \right\} \quad (5.3.13)$$

---

<sup>3</sup>One could calculate the difference of the left endpoints (5.3.11) directly, but we found it easier this way.



where  $f(m, n)$  could have elements smaller than  $d - 2r + 2$  is finite and have  $r - 2$  elements. We divide the problem into two cases:

- a) when  $\sqrt{d^2 - 4(r - 1)(g - 1)} > 2r - 2$  and  $n \geq r - 1$ ,
- b) when  $\sqrt{d^2 - 4(r - 1)(g - 1)} \leq 2r - 2$  and  $n \geq r - 1$ .

Consider the situation in a); First observe that

$$f(-bn, n) = n(2g - 2 - bd) \geq 2g - 2 - bd$$

and

$$2g - 2 - bd > d - 2r + 2 \Leftrightarrow 2r - 2 < \sqrt{d^2 - 4(r - 1)(g - 1)} < d - 2r + 2.$$

We are left with the case where  $\sqrt{d^2 - 4(r - 1)(g - 1)} \geq d - 2r + 2$ .

$$\begin{aligned} f\left(\left\lceil \frac{2 - nd}{2r - 2} \right\rceil, n\right) &\geq f\left(\frac{2 - nd}{2r - 2}, n\right) \\ &= \frac{(n^2 - n)(d^2 - 4(r - 1)(g - 1)) + 2d - 4}{2r - 2} \\ &\geq \frac{r - 2}{2}(d^2 - 4(r - 1)(g - 1))^2 + \frac{d - 2}{r - 1} \\ &\geq \frac{r - 2}{2}(d - 2r + 2)^2 + \frac{d - 2}{r - 1} \\ &> d - 2r + 2. \end{aligned}$$

This is obvious when  $r \geq 4$ . But when  $r = 3$  get

$$\begin{aligned} \frac{1}{2}(d - 4)^2 + \frac{d - 2}{2} &= \frac{(d - 4)^2 + d - 4}{2} + 1 \\ &= (d - 4) \left[ \frac{d - 4}{2} + \frac{1}{2} \right] + 1 \\ &\geq d - 4, \end{aligned}$$

because  $d \geq 12$ , so that  $\frac{d-4}{2} + \frac{1}{2} > 1$ . This shows case a).

Consider situation b);

$$\max(\mathcal{C}) = f(\lceil -bn \rceil, n),$$

since

$$\frac{2 - nd}{2r - 2} < \frac{n\sqrt{d^2 - 4(r-1)(g-1)} - nd}{2r - 2}$$

and  $f(m, n)$  is increasing on  $\left(\frac{2-nd}{2r-2}, \frac{d(1-2n)}{4(r-1)}\right]$ .

Now we will show that if  $n \geq r - 1$ , then

$$\begin{aligned} f(\lceil -bn \rceil, n) &\geq f(-bn, n) = \frac{d\sqrt{d^2 - 4(r-1)(g-1)} - (d^2 - 4(r-1)(g-1))}{2r - 2}n \\ &\geq d - 2r + 2. \end{aligned}$$

A small calculation gives that

$$\begin{aligned} f(-bn, n) &= \frac{d\sqrt{d^2 - 4(r-1)(g-1)} - (d^2 - 4(r-1)(g-1))}{2r - 2}n \\ &\geq \frac{d\sqrt{d^2 - 4(r-1)(g-1)} - (d^2 - 4(r-1)(g-1))}{2} \\ &= \frac{1}{2} \left[ d - \sqrt{d^2 - 4(r-1)(g-1)} \right] \sqrt{d^2 - 4(r-1)(g-1)} \end{aligned} \tag{5.3.14}$$

Using that

$$2r - 2 > \sqrt{d^2 - 4(r-1)(g-1)} > 2\sqrt{r-1},$$

we see that formula (5.3.14) is bigger than  $d - 2r + 2$ , i.e.,

$$\begin{aligned} f(-bn, n) &\geq \frac{1}{2} [d - \sqrt{d^2 - 4(r-1)(g-1)}] \sqrt{d^2 - 4(r-1)(g-1)} \\ &> \frac{1}{2} [d - (2r - 2)] 2\sqrt{r-1} \\ &> d - 2r + 2, \end{aligned}$$

since  $r \geq 3$ . Now we have showed that the set (5.3.13) for which  $f(m, n)$  can be less than  $d - 2r + 2$  is finite and have  $r - 2$  elements. We are now left with the interval  $0 < n < r - 1$ . We treat this case in the same way as we did with  $n < 0$ -case. From the inequality  $m \geq -bn$  we obtain

$$m \geq \begin{cases} \frac{\lceil 2n\sqrt{r-1} \rceil - nd}{2r-2} & , \text{ if } \sqrt{r-1} \notin \mathbb{Z} \\ \frac{\lceil 2n\sqrt{r-1} \rceil + 1 - nd}{2r-2} & , \text{ if } \sqrt{r-1} \in \mathbb{Z}. \end{cases}$$

As done before put  $\epsilon = 0$  if  $\sqrt{r-1} \notin \mathbb{Z}$  and  $\epsilon = 1$  if not.

The only case left is where  $n = 0$ , and since  $f(m, 0) = -(2r-2)m^2 + md$ . Clearly  $f(m, 0) \geq f(1, 0) = d - 2r + 2$  for all  $m$  complying with i), ii) and iii).

Now we must show that the candidates we have which can give lower gonality than the expected gonality is in fact inside the boundaries i), ii), iii) on p. 51.

i) is always fulfilled.

We show that ii) is fulfilled simultaneously for positive  $n$  and negative  $n$ . We are done if we can show that

$$\frac{2|n|\sqrt{r-1} + \epsilon - nd}{2r+2} + 1,$$

satisfies ii). Insert the expression above into ii), we then get

$$2 < 2|n|\sqrt{r-1} + \epsilon + 2r - 2 < d - 2.$$

The first inequality is obvious. To show the second inequality, we use  $\sqrt{r-1} < \frac{r-1}{2}$ ,  $|n| < r-1$ ,  $\epsilon$  is either 0 or 1 and the assumption  $d \geq r^2 + r$ , so

$$\begin{aligned} 2|n|\sqrt{r-1} + \epsilon + 2r - 2 &< (r-1)^2 + \epsilon + 2r - 2 \\ &\leq r^2 \\ &< r^2 + r - 2 \\ &\leq d - 2. \end{aligned}$$

The third boundary condition may not be fulfilled for some  $d$ ,  $g$  and  $r$ , therefore we have to take this into account.  $\square$

We will now see an example of a divisor on a  $K3$  surface satisfying the assumptions in the theorem above, which induces lower gonality than the expected one.

**Example 5.2.** Let  $r = 3$ ,  $d \geq 12$  and  $g \geq 0$  be integers and let  $C$  be the smooth curve on a quartic  $K3$  surface in  $\mathbb{P}^3$ . We allow rational and elliptic curves. For example we look at the divisor  $D \sim 8H - C$ , which lies on  $S$ . This divisor must lie in the family of effective divisors given in Lemma 4.8. From Theorem 5.1, we get

$$\left\lceil \frac{d+3}{4} \right\rceil = m \text{ implies } 4m - 7 < d \leq 4m - 3.$$

Since  $m = 8$ ,  $d$  can take the values 26, 27, 28 and 29. We must<sup>4</sup> choose  $d = 29$ . When  $r = 3$ , the inequality  $D^2 \geq 0$  is

$$\begin{aligned} D^2 &= 2m^2 + mnd + n^2(g - 1) \\ &= -105 + g \\ &\geq 0, \end{aligned}$$

so we must have  $g \geq 105$ . But the relation  $d^2 > 8g$  gives

$$d^2 = 841 > 8g \geq 840,$$

So the only possibility is  $g = 105$ . Now we clearly see that  $D^2 = 0$  and

$$D.C = 8H.C - C^2 = 8 \cdot 29 - 2 \cdot 104 = 24 \leq 104.$$

So  $D \in \mathcal{A}$ . We have found that  $d = 29$ ,  $g = 105$ . Now we use the standard formula (5.3.1) for calculating gonality,

$$\begin{aligned} f(8, -1) &= -4 \cdot 8^2 + 8(29 - 2(-1)29) + (-1 - (-1)^2)(2 \cdot 105 - 2) \\ &= 24. \end{aligned}$$

Comparing this number with the expected gonality (7.1.1),  $\min\{25, 54\} = 25$ . It is important to notice that this divisor is inside the boundary on page 51. Thus, we have found a curve on a  $K3$  surface which gives a number less than the expected gonality.

There are many curves on a  $K3$  surface, satisfying the assumptions in theorem above, which gives that the number  $f(m, n)$  is less than the expected gonality. Let us still be in  $\mathbb{P}^3$  and with  $n = -1$ , as we did in the example above. The following table shows development when  $m$  increases:

$D \sim mH - C$	$d$	$g$	$f(m, -1)$	$d - 4$	$\lfloor \frac{g+3}{2} \rfloor$
$8H - C$	29	105	24	25	54
$9H - C$	33	136	27	29	69
$10H - C$	37	171	30	33	87
$11H - C$	41	210	33	37	106

Note that the first divisor is the divisor in Example 5.2 and an easy calculation shows that the values of  $d, g$  and  $r$  satisfies the boundary conditions on p. 51.

<sup>4</sup>An easy calculation shows that if  $d = 26, 27$  or  $28$ , then  $f(8, -1) > d - 4$ .

**Corollary 5.3.** *Let  $r = 3, 4, 5$ ,  $d, g$  and  $f(m, n)$  be as in Theorem 5.1 and let  $C$  be a smooth curve on a  $K3$  surface  $S \subseteq \mathbb{P}^r$ . Then*

$$\text{gon}(C) = \min \left\{ f \left( \left\lceil \frac{d+r}{2r-2} \right\rceil, -1 \right), f \left( \left\lceil \frac{r-d}{2r-2} \right\rceil, 1 \right), d-2r+2, \left\lfloor \frac{g+3}{2} \right\rfloor \right\},$$

and  $m := \left\lceil \frac{d+r}{2r-2} \right\rceil$  must satisfy  $md - 3(g-1) \leq 0$ .

*Proof.* The proof is just some easy calculations.

$r = 3$ : This case follows directly from Theorem 5.1.

$r = 4$ : From Theorem 5.1 we have

$$N_4 = \min \left\{ f \left( \left\lceil \frac{d+4}{6} \right\rceil, -1 \right), f \left( \left\lceil \frac{4-d}{6} \right\rceil, 1 \right), f \left( \left\lceil \frac{2d+7}{6} \right\rceil, -2 \right), f \left( \left\lceil \frac{7-2d}{6} \right\rceil, 2 \right) \right\}.$$

So we need only to look at the last 2 elements.

$$\begin{aligned} f \left( \left\lceil \frac{2d+7}{6} \right\rceil, -2 \right) &\geq f \left( \frac{2d+7}{6}, 2 \right) \\ &= \frac{6(d^2 - 6g)}{6} + \frac{7d}{6} + \frac{23}{6} \\ &> d - 6 \end{aligned}$$

and

$$\begin{aligned} f \left( \left\lceil \frac{7-2d}{6} \right\rceil, 2 \right) &\geq f \left( \frac{7-2d}{6}, 2 \right) \\ &= \frac{2(d^2 - 12g)}{6} + \frac{7d}{6} - \frac{25}{6} \\ &> d - 6 \end{aligned}$$

$r = 5$ : Using Theorem 5.1 again, we obtain

$$\begin{aligned} N_5 = \min \{ & f \left( \left\lceil \frac{d+5}{8} \right\rceil, -1 \right), f \left( \left\lceil \frac{5-d}{8} \right\rceil, 1 \right), f \left( \left\lceil \frac{2d+9}{8} \right\rceil, -2 \right), f \left( \left\lceil \frac{3d+13}{8} \right\rceil, -3 \right), \\ & f \left( \left\lceil \frac{9-2d}{8} \right\rceil, 2 \right), f \left( \left\lceil \frac{13-3d}{8} \right\rceil, 3 \right) \}. \end{aligned}$$

The last 4 elements gives:

$$\begin{aligned} f\left(\left\lceil \frac{2d+9}{8} \right\rceil, -2\right) &\geq f\left(\frac{2d+9}{8}, -2\right) \\ &= \frac{6(d^2 - 16g)}{8} + \frac{9d}{8} + \frac{15}{8} \\ &> d - 8. \end{aligned}$$

$$\begin{aligned} f\left(\left\lceil \frac{9-2d}{8} \right\rceil, 2\right) &\geq f\left(\frac{9-2d}{8}, 2\right) \\ &= \frac{2(d^2 - 16g)}{8} + \frac{9d}{8} - \frac{49}{8} \\ &> \frac{9d}{8} - \frac{49}{8} \\ &\geq d - 8. \end{aligned}$$

This inequality holds for all  $d \geq 1$ .

$$\begin{aligned} f\left(\left\lceil \frac{3d+13}{8} \right\rceil, -3\right) &\geq f\left(\frac{3d+13}{8}, -3\right) \\ &= \frac{12(d^2 - 16g)}{8} + \frac{13d}{8} + \frac{23}{8} \\ &> d - 8, \end{aligned}$$

For the last element, we get

$$\begin{aligned} f\left(\left\lceil \frac{13-3d}{8} \right\rceil, 3\right) &\geq f\left(\frac{13-3d}{8}, 3\right) \\ &= \frac{2(d^2 - 16g)}{8} + \frac{13d}{8} - \frac{73}{8} \\ &\geq \frac{13d}{8} - \frac{71}{8} \\ &\geq d - 8, \end{aligned}$$

if  $5d \geq 7$ . But  $d \geq 30$ , so this is also true.

When  $n = 1$ , we want to show that  $m = \left\lceil \frac{r-d}{2r-2} \right\rceil$  satisfies the boundary condition  $md + g - 1 \leq 0$ . Inserting,

$$d \left\lceil \frac{r-d}{2r-2} \right\rceil + g - 1 \leq 0 \text{ implies } g - 1 \leq d \left\lfloor \frac{d-r}{2r-2} \right\rfloor,$$

Using  $d \leq g - 1$ ,  $d^2 > 4(r - 1)g$  and a straightforward calculation

$$\begin{aligned} d \left\lfloor \frac{d - r}{2r - 2} \right\rfloor &\geq \frac{d(d - r)}{2r - 2} \\ &> \frac{4(r - 1)g - dr}{2r - 2} = 2g - \frac{dr}{2r - 2} \\ &\geq 2g - \frac{(g - 1)r}{2(r - 1)} \\ &\geq 2g - (g - 1) = g + 1. \end{aligned}$$

This is of course greater than  $g - 1$ , so  $\left\lceil \frac{r - d}{2r - 2} \right\rceil$  satisfies the boundary condition.  $\square$

There are two things we will discuss, the first thing are the numbers  $N_3$ ,  $N_4$  and  $N_5$  and the second is the boundary condition  $md - (2n - 1)(g - 1) \leq 0$ . We need only to look at the case where  $r = 3$ .

Below we will show that when  $d$  and  $g$  are not fixed,  $N_3$  is equal to the first element if  $\frac{d+3}{4}$  is an integer and equal to the second if  $\frac{3-d}{4}$  is an integer and we will see that the integer  $m := \frac{d+3}{4}$  does satisfy the boundary condition  $md - 3(g - 1) \leq 0$ .

By Corollary 5.3

$$N_3 = \min \left\{ f \left( \left\lceil \frac{d+3}{4} \right\rceil, -1 \right), f \left( \left\lceil \frac{3-d}{4} \right\rceil, 1 \right) \right\}, \quad (5.3.15)$$

where  $m := \left\lceil \frac{d+3}{4} \right\rceil$  must satisfy the boundary condition  $md - 3(g - 1) \leq 0$ . Write

$$\left\lceil \frac{d+3}{4} \right\rceil = x \Rightarrow \frac{d+3}{4} + t = x$$

and

$$\left\lceil \frac{3-d}{4} \right\rceil = y \Rightarrow \frac{3-d}{4} + s = y,$$

for  $t, s \in [0, 1)$ . It is easy to see that  $t$  and  $s$  can only take the values  $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ , with the relations

$$t = 0 \Leftrightarrow s = \frac{1}{2}, t = \frac{1}{4} \Leftrightarrow s = \frac{3}{4}, t = \frac{1}{2} \Leftrightarrow s = 0 \text{ and } t = \frac{3}{4} \Leftrightarrow s = \frac{1}{4}$$

Now we write out the formulas

$$f \left( \left\lceil \frac{d+3}{4} \right\rceil, -1 \right) = \frac{2(d^2 - 8g)}{4} + \frac{(3 + 4t)d}{4} - 6t - 4t^2 + \frac{7}{4} \quad (5.3.16)$$

and

$$f\left(\left\lceil \frac{3-d}{4} \right\rceil, 1\right) = \frac{(3+4s)d}{4} - 6s - 4s^2 - \frac{9}{4}. \quad (5.3.17)$$

An easy calculation shows that for  $s, t$ -values  $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ , formulas (5.3.16) and (5.3.17) gives

$$f\left(\left\lceil \frac{d+3}{4} \right\rceil, -1\right) \geq d-4 \text{ and } f\left(\left\lceil \frac{3-d}{4} \right\rceil, 1\right) \geq d-4,$$

since  $d \geq 12$ . We are only left with  $s, t = 0$ , that is, when  $\frac{d+3}{4} \in \mathbb{Z}$  and  $\frac{3-d}{4} \in \mathbb{Z}$ .

When  $t = 0$  the formula (5.3.16) gives

$$f\left(\frac{d+3}{4}, -1\right) = \frac{2(d^2 - 8g)}{4} + \frac{3d}{4} + \frac{7}{4}.$$

One can show that  $d^2 - 8g = 1$ . The boundary condition i) on page 51 gives

$$g \geq \frac{d^2}{8} - \frac{1}{8}$$

so that

$$1 \leq d^2 - 8g \leq 1,$$

hence the equality.  $f\left(\frac{d+3}{4}, -1\right) = \frac{3d}{4} + \frac{9}{4} < d-4$  if  $d \geq 26$ .

Similarly when  $s = 0$ , the formula (5.3.17) gives

$$f\left(\frac{3-d}{4}, 1\right) = \frac{3d}{4} - \frac{9}{4} < d-4$$

when  $d \geq 8$ . Finally we will that  $\frac{d+3}{4}$  satisfies  $md - 3(g-1) \leq 0$ .

$$\begin{aligned} \frac{3+d}{4}d - 3(g-1) &= \frac{3d - 4(g-1) + 9}{4} \\ &\leq \frac{10-g}{4} \leq 0, \end{aligned}$$

if  $g \leq 10$ , but  $d \geq 12$  and  $d \leq g-1$ , so  $m = \frac{d+3}{4}$  satisfy  $md - 3(g-1) \leq 0$ . This shows that  $N_3$  (5.3.15) can be both, when  $d$  and  $g$  varies in the domain i), ii) and iii) on p. 51. We can do the same in the cases where  $r = 4$  or  $5$ , it is the same calculations, but with different numbers, so we end our discussion here.



# Chapter 6

## The Picard Group of Rank 3

In the last section we calculated gonality of curves on  $K3$  surfaces with Picard group of rank 2. Now we will take one step further, and add one more generator to the Picard group. We found out that it would have been very difficult calculate the gonality in general by using the same numerical techniques, as we did in Theorem 5.1. This led to some restrictions, so that the results became a bit weaker than we hoped for.

### 6.1 Existence of an Algebraic $K3$ Surface

We need to show under what circumstances there exists an algebraic  $K3$  surface with Picard group of rank 3. The existence is showed by using Proposition 4.1.

**Proposition 6.1.** *Let  $r \geq 3$ ,  $d > 0$  and  $g \geq 0$  be integers. Then there exists an algebraic  $K3$  surface  $S$  with Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$  with  $H$  nef and with the following intersection numbers  $H^2 = 2r - 2$ ,  $C^2 = 2g - 2$ ,  $E^2 = 2\beta$ ,  $H.C = d$ ,  $H.E = \alpha$  and  $C.E = k$  if and only if the following conditions are satisfied*

- i)  $C := d^2 - 4(r - 1)(g - 1) + k^2 - 4\beta(r - 1) - 4\beta(g - 1) - \alpha^2 > 0$ ,
- ii)  $D := 4\beta(r - 1)(g - 1) - k^2(r - 1) - \beta d^2 + dk\alpha - (g - 1)\alpha^2 > 0$ ,
- iii)  $x_- := \frac{2B - \sqrt{2B^2 + 3C}}{3}$  and  $B := g + \beta + r - 2$  gives that the third degree polynomial,  $-x^3 + 2Bx^2 + Cx + 2D$ , valuated at  $x_-$  is less than 0.

*Proof.* First we show existence of a  $K3$  surface  $S$  with Picard group given by  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ . We do this by showing that this lattice is even with signature  $(1, 2)$ . Consider the intersection matrix,

$$L = \begin{pmatrix} H^2 & H.C & H.E \\ C.H & C^2 & C.E \\ E.H & E.C & E^2 \end{pmatrix} = \begin{pmatrix} 2r-2 & d & \alpha \\ d & 2g-2 & k \\ \alpha & k & 2\beta \end{pmatrix} \quad (6.1.1)$$

We must show that the determinant is even and has one positive eigenvalue and two negative eigenvalues, by Theorem 4.1. Elementary linear algebra shows that

$$\begin{aligned} f(x) = \det(L - xI) &= -x^3 + 2[g + \beta + r - 2]x^2 \\ &+ [d^2 - 4(r-1)(g-1) + k^2 - 4\beta(r-1) - 4\beta(g-1) - \alpha^2]x \\ &+ 2[4\beta(r-1)(g-1) - k^2(r-1) - \beta d^2 + dk\alpha - (g-1)\alpha^2]. \end{aligned}$$

Notice that the last summand is just  $\det L = \text{disc}(H, C, E)$  and is clearly even. To make it simple we let the polynomial above be expressed as

$$f(x) = -x^3 + 2Bx^2 + Cx + 2D,$$

where the constants are

$$B = g + \beta + r - 2, \quad (6.1.2)$$

$$C = d^2 - 4(r-1)(g-1) + k^2 - 4\beta(r-1) - 4\beta(g-1) - \alpha^2, \quad (6.1.3)$$

$$D = 4\beta(r-1)(g-1) - k^2(r-1) - \beta d^2 + dk\alpha - (g-1)\alpha^2. \quad (6.1.4)$$

We are only interested in the sign of the eigenvalues of the intersection matrix  $L$ . We need to put some requirements on the constants. But first look at the derivative of  $f(x)$ ,

$$f'(x) = -3x^2 + 4Bx + C.$$

We find the extremal points

$$x = \frac{2B \mp \sqrt{4B^2 + 3C}}{3} = x_{\mp}.$$

Clearly  $B \geq 0$ . Moreover, we have that  $f''(x) > 0$  when  $x < \frac{2B}{3}$  and non-positive otherwise. We must have two zeros of  $f(x)$  when  $x < 0$ . Therefore, we must require that  $C > 0$ ,  $D > 0$  and that  $f(x_-) < 0$ . Now Theorem 4.1 says that there exists an algebraic  $K3$  surface with Picard group

$\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ . It remains to show that  $H$  can be chosen nef. We will follow [BPVdV84], p. 238-243.

$$\Delta := \{\Gamma \in \text{Pic}(S) \mid \Gamma^2 = -2\}$$

and consider the Picard-Lefschetz reflection

$$\pi_\Gamma : \text{Pic}(S) \longrightarrow \text{Pic}(S) \tag{6.1.5}$$

$$D \longmapsto D + (D.\Gamma)\Gamma, \tag{6.1.6}$$

which clearly preserves intersections between divisors, i.e.

$$\begin{aligned} \pi_\Gamma(D).\pi_\Gamma(E) &= [D + (D.\Gamma)\Gamma].[E + (E.\Gamma)\Gamma] \\ &= D.E + 2(D.\Gamma)(E.\Gamma) + \Gamma^2(D.\Gamma)(E.\Gamma) \\ &= D.E \end{aligned}$$

Let

$$\mathcal{C}_S = \{D \in \text{Pic}(S) \mid D \text{ is effective and } D^2 > 0\}$$

be the *positive cone* of  $S$  and

$$\mathcal{C}_S^+ = \{D \in \mathcal{C}_S \mid \Gamma.D > 0 \text{ for all } \Gamma \in \Delta\}$$

be the *Kähler cone*, with

$$\overline{\mathcal{C}_S^+} = \{D \in \mathcal{C}_S \mid \Gamma.D \geq 0 \text{ for all } \Gamma \in \Delta\}$$

its closure, which is the *big and nef cone* of  $S$ , since it consists of all divisors in  $S$  that are nef and big.

Proposition VIII, 3.9 in [BPVdV84] states that the Picard-Lefschetz reflections of  $S$ , the set  $\{\pi_\Gamma\}_{\Gamma \in \Delta}$ , leave invariant  $\mathcal{C}_S$  and any orbit in  $\mathcal{C}_S$  of the group generated by  $\{\pi_\Gamma\}_{\Gamma \in \Delta}$  meets  $\overline{\mathcal{C}_S^+}$  in exactly one point.

Since  $H \in \mathcal{C}_S$  ( $H^2 > 0$ , either  $|H|$  or  $|-H|$  contains an effective member. If it is  $|-H|$  we can change  $H$  with  $-H$ ,  $C$  with  $-C$  and  $E$  with  $-E$ ), we can make Picard-Lefschetz reflections on  $\text{Pic}(S)$  until we get new divisors  $H'$ ,  $C'$  and  $E'$  with the same intersection matrix as  $H$ ,  $C$  and  $E$ , such that  $H'$  is nef. We only have to show that  $\mathbb{Z}H' \oplus \mathbb{Z}C' \oplus \mathbb{Z}E' = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ .

For each reflection  $\pi_\Gamma$ , we have

$$\begin{aligned} H &\mapsto H + (H.\Gamma)\Gamma := H_\Gamma \\ C &\mapsto C + (C.\Gamma)\Gamma := C_\Gamma \\ E &\mapsto E + (E.\Gamma)\Gamma := E_\Gamma. \end{aligned}$$

This means that  $\mathbb{Z}H_\Gamma \oplus \mathbb{Z}C_\Gamma \oplus \mathbb{Z}E_\Gamma \subseteq \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , since  $\Gamma \in \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ . An easy calculation shows that  $\pi_\Gamma^2$  is the identity map on  $\text{Pic}(S)$ , so we get the other inequality. This is true for all  $\Gamma \in \text{Pic}(S)$ , so we have proved that  $\mathbb{Z}H' \oplus \mathbb{Z}C' \oplus \mathbb{Z}E' = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$  and we can assume that  $H$  is nef.  $\square$

*Remark 6.2.* From now on, we will restrict ourself to the case where  $\beta = 0$ , i.e.  $E^2 = 0$ . This will make the calculations a lot easier. Another note, is that  $\alpha$  cannot be chosen arbitrary. We can see this from the formulas for  $C$  and  $D$ , when we let  $\beta = 0$ , in the theorem above. We can choose it arbitrary within the interval where there exist an algebraic  $K3$  surface.

## 6.2 Existence of Hyperelliptic and Tetragonal Curves on $S$

The next thing we need to show is the very ampleness of  $H$ , such that we get an embedding into a projective space, and the nefness of the generators  $C$  and  $E$ . This can be shown using Lemma 4.4. In general, this turned out to be a hard numerical problem. Therefore we made some restrictions on  $(d, g, r)$ , which eliminated the rational curves on these  $K3$  surfaces.

In the proposition below, we will show that under a certain parity of  $(d, g)$  we can find a hyperelliptic curve and a tetragonal curve on  $S$ . In the next section we will look at a more general result. Even though the proposition below is generalized in the next section, we decided to keep it in the thesis, because it shows the very ampleness of  $H$  and the proof is different from that in the next section.

**Proposition 6.3.** *Let  $r \geq 3$ ,  $d > 0$  and  $g > 0$  are integers such that  $r$  and  $g$  are odd and  $d$  is divisible by 4. Then there exist a  $K3$  surface  $S$  of degree  $2r - 2$  in  $\mathbb{P}^r$  with Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$  containing a nondegenerate hyperelliptic curve (resp. tetragonal curve) of degree  $d$  and genus  $g$ . The intersection numbers between the generators in the Picard group are  $H^2 = 2r - 2$ ,  $C^2 = 2g - 2$ ,  $E^2 = 0$ ,  $H.C = d$ ,  $H.E = \alpha$ , where  $\alpha$  is divisible by 4 and  $C.E = 2$  (resp. 4).*

*Proof.* By Proposition 6.1 there exist a  $K3$  surface  $S$  with Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , the intersection numbers are  $H^2 = 2r - 2$ ,  $C^2 = 2g - 2$ ,  $E^2 = 0$ ,  $H.C = d$ ,  $H.E = \alpha$  and  $C.E = 2$  (resp. 4). Let  $D \in \text{Pic}(S)$

be a divisor on  $S$ , then  $D$  can be written as  $xH + yC + zE$ . We need to embed  $S$  in a projective space. We will make use of Lemma 4.4 b) i) and iii). We get that  $H^2 = 2r - 2 \geq 4$  and we have

$$\frac{D^2}{2} = (r-1)x^2 + dxy + \alpha xz + (g-1)y^2 + kyz \text{ and } D.H = (2r-2)x + dy + \alpha z,$$

where  $k$  is either 2 or 4. Let us recall i) and iii) in Lemma 4.4

*Claim 1* : There is no divisor  $D \in \text{Pic}(S)$  such that  $D^2 = 0$  and  $D.H = 1, 2$ .

*Proof* We have to show that  $D.H = (2r-2)x + dy + \alpha z \neq 1, 2$ . Since  $r$  is odd and  $d$  is even by assumption, we can choose  $\alpha$  to be even. This means that  $D.H$  can never be equal to 1. For the case where  $D.H = 2$ , we obtain

$$\frac{D.H}{2} = (r-1)x + \frac{d}{2}y + \frac{\alpha}{2}z. \quad (6.2.1)$$

Since  $d$  is divisible by 4, and we have chosen  $\alpha$  to be even. Since  $\alpha$  can be chosen arbitrary, in some sense, we choose it to be divisible by 4. This means that  $D.H \neq 2$ .

*Claim 2* : There is no divisor  $D \in \text{Pic}(S)$  such that  $D^2 = -2$  and  $D.H = 0$ .

*Proof* We see immediately that  $D^2 \neq -2$ , since

$$\frac{D^2}{2} = (r-1)x^2 + (g-1)y^2 + dxy + \alpha xz + kyz \neq -1$$

for  $r, g$  odd integers,  $d$  and  $\alpha$  even integers and  $k$  is either 2 or 4.

These two situations never occurs, so we conclude that the hyperplane section  $H$  is very ample, according to Lemma 4.4. Note also that  $C$  is nef and  $|C|$  is base-point free, this follows immediately from the assumption that  $S$  does not contain  $-2$ -curves, by Lemma 4.4 a). As in the rank 2 case, we have the family of effective divisors given in Lemma 4.8.

The situation in the Donagi-Morrison example and the generalized ELMS examples cannot occur, thus we get that the gonality can be calculated using  $\text{gon}(C) = \min\{D.C - D^2 \mid D \in \mathcal{A}\}$ , by Lemma 4.8.  $D \in \text{Pic}(S)$ , so we write  $D \sim xH + yC + zE$ .  $D.C = (xH + yC + zE).C = dx + 2(g-1)y + kz$ , we

express  $D.C - D^2$  in terms of  $D.C$ ,

$$\begin{aligned}
D.C - D^2 &= D.C - (xH + yC + zE).(xH + yC + zE) \\
&= D.C - [2(r-1)x^2 + 2xyd + 2\alpha xz + 2(g-1)y^2 + 2kyz] \\
&= D.C - 2(r-1)x^2 - 2y[xd + 2(g-1)y + kz] + 2(g-1)y^2 - 2\alpha xz \\
&= D.C - 2(r-1)x^2 - 2yD.C + 2(g-1)y^2 - 2\alpha xz \\
&= -2(r-1)x^2 + 2(g-1)y^2 - 2\alpha xz + (1-2y)D.C.
\end{aligned}$$

The case  $\text{gon}(C) = 1$  does not occur, since  $D.C - D^2$  is always an even integer by the assumption of  $r, d$  and  $g$ . By the Hodge Index theorem, we have

$$\begin{vmatrix} C^2 & D.C \\ C.D & D^2 \end{vmatrix} = C^2 D^2 - (D.C)^2 \leq 0$$

which implies

$$2(g-1) \leq \frac{(D.C)^2}{D^2}$$

if  $D^2 > 0$ . For curves with low gonality, we may assume  $D^2 = 0$ , since we don't want small upper boundary on  $g$ .

i) In the case  $k = 2$ , we have  $d$  divisible by 4 and  $g$  odd. Hence,

$$\begin{aligned}
\text{gon}(C) &= \min\{D.C\} \\
&= \min\{dx + 2(g-1)y + 2z\} \\
&= 2.
\end{aligned}$$

This shows that  $C$  is hyperelliptic if  $k = 2$ .

ii) In the case  $k = 4$ , we  $d$  divisible by 4 and  $g$  odd. We need to eliminate the case where  $D.C = 2$ .

$$\frac{D.C}{2} = \frac{d}{2}x + (g-1)y + 2z \neq 1.$$

Therefore  $\text{gon}(C) = 4$  and  $C$  is tetragonal.

□

### 6.3 Existence of $k$ -gonal Curves on $S$

In this section we will study the gonality of curves in  $\mathbb{P}^r$  with a base point free complete linear system,  $\mathfrak{g}_k^1$ . We will try to answer the question: What condition can we put on a curve  $C$ , such that  $C$  contains  $\mathfrak{g}_k^1$ , but no  $\mathfrak{g}_l^1$  for  $l < k$ , equivalently, what conditions can we put on a curve  $C$  such that  $\text{gon}(C) = k$ ?

In the next example will be considered as a motivation for Theorem 6.5 below.

**Example 6.4.** Let  $C$  be a curve of genus  $g \geq 3$  which has a 3-to-1 map to  $\mathbb{P}^1$  and has a 2-to-1 map to  $\mathbb{P}^1$ . We will show that this is impossible. Let  $f : C \xrightarrow{3:1} \mathbb{P}^1$  be the 3-to-1 map and  $g : C \xrightarrow{2:1} \mathbb{P}^1$  be the 2-to-1 map. Let the arithmetic genus  $p_a(C) = g$ . The map

$$\begin{aligned} (f, g) : C &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ C &\longmapsto C_0. \end{aligned}$$

is birational.  $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}l_1 \oplus \mathbb{Z}l_2$ , where  $l_{1|C}$  and  $l_{2|C}$  are the fibers in the pencils  $\mathfrak{g}_2^1$  and  $\mathfrak{g}_3^1$  at a given point in  $\mathbb{P}^1$ , respectively. Notice that  $\mathfrak{g}_2^1$  and  $\mathfrak{g}_3^1$  are not composed, since 2 and 3 are relatively prime. Since  $C_0 = \alpha l_{1|C} + \beta l_{2|C}$  for some  $\alpha, \beta \in \mathbb{Z}$ . Let us find the value of these integers. Since  $l_{1|C}$  is a fiber in  $\mathfrak{g}_2^1$ , we have that  $l_{1|C} \cdot C_0 = 2$ . Hence

$$2 = l_{1|C} \cdot C_0 = l_{1|C} \cdot (\alpha l_{1|C} + \beta l_{2|C}) = 0 + \beta,$$

which implies that  $\beta = 2$ . The same reasoning gives that  $\alpha = 3$ , so our divisor can be written as  $C_0 = 3l_{1|C} + 2l_{2|C}$ . The canonical divisor on  $\mathbb{P}^1 \times \mathbb{P}^1$  is  $K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2l_{1|C} - 2l_{2|C}$ . Applying the adjunction formula (2.4.5),

$$\begin{aligned} 2p_a(C_0) - 2 &= C_0 \cdot (C_0 + K_{\mathbb{P}^1 \times \mathbb{P}^1}) = (3l_{1|C} + 2l_{2|C}) \cdot (3l_{1|C} + 2l_{2|C} - 2l_{1|C} - 2l_{2|C}) \\ &= (3l_{1|C} + 2l_{2|C}) \cdot l_{1|C} \\ &= 2, \end{aligned}$$

thus  $p_a(C_0) = 2$ . Moreover, by Proposition 2.1 we have the following inequality

$$p_g(C) = p_a(C) \leq p_a(C_0) = 2,$$

which is a contradiction since  $p_a(C) = g \geq 3$ .

**Theorem 6.5.** *Let  $C$  be a curve with a complete base point free linear system  $\mathfrak{g}_k^1$ . If the genus of  $C$  is  $g > (k-1)(k-2)$ , then  $\text{gon}(C) = k$ .*

*Proof.* Define maps  $f : C \xrightarrow{a:1} \mathbb{P}^1$  and  $g : C \xrightarrow{k:1} \mathbb{P}^1$ . Assume  $\text{gon}(C) = a < k$ . Let  $C$  be a curve with  $p_a(C) = g$ . Since the pencil  $\mathfrak{g}_k^1$  is not a multiple of  $\mathfrak{g}_a^1$ , that is,  $\mathfrak{g}_k^1 \neq l\mathfrak{g}_a^1$  for some  $a \in \mathbb{Z}$ , because  $\mathfrak{g}_k^1$  is a complete base point free linear system, the map  $(f, g) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is birational. We know that the canonical divisor on  $\mathbb{P}^1 \times \mathbb{P}^1$  is equal to  $-2l_1 - 2l_2$  for fibers  $l_1 = f^{-1}(m)$  and  $l_2 = g^{-1}(n)$ . By the adjunction formula, we obtain

$$\begin{aligned} 2p_a(C_0) - 2 &= C_0 \cdot (C_0 + K_{\mathbb{P}^1 \times \mathbb{P}^1}) = (kl_1 + al_2) \cdot (kl_1 + al_2 - 2l_1 - 2l_2) \\ &= 2((a-1)k - a). \end{aligned}$$

We know that the geometric genus is always less than or equal to the arithmetic genus, this gives the following relation

$$p_g(C_0) = p_a(C) \leq p_a(C_0) = (a-1)k - a + 1.$$

Since  $a \leq k-1$  implies that  $(a-1)k - a + 1 \leq (k-1)k - (k-1) + 1 = (k-1)(k-2)$ , but  $g > (k-1)(k-2)$  and thus a contradiction.  $\square$

**Corollary 6.6.** *Let  $r, g$  be odd integers and  $d$  divisible by 4. Then there exists a K3 surface  $S$  of degree  $2r-2$  in  $\mathbb{P}^r$  with Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , where  $H = \mathcal{O}_S(1)$ , containing a nondegenerate  $k$ -gonal curve of degree  $d$  and genus  $g > (k-1)(k-2)$ . The intersection numbers between the generators in the Picard group are  $H^2 = 2r-2$ ,  $C^2 = 2g-2$ ,  $E^2 = 0$ ,  $H \cdot E = \alpha$ , where  $\alpha$  is divisible by 4,  $C \cdot E = k$ .*

*Proof.* The existence part of  $S$  is taken care of by Theorem 6.1. Note that under conditions on the triplet  $(d, g, r)$ ,  $S$  does not contain  $-2$ -curves (see the proof of Proposition 6.3). The embedding of  $S$  into  $\mathbb{P}^r$ , is taken care of in the proof of Proposition 6.3. Let  $|E|_C$  be a linear system on  $C$ . This linear system is a  $\mathfrak{g}_k^1$ , since  $\deg \mathcal{O}_C(E) = E \cdot C = k$ , and we want to show it is complete and base point free. We look at the standard exact sequence

$$0 \rightarrow \mathcal{O}_S(E - C) \rightarrow \mathcal{O}_S(E) \rightarrow \mathcal{O}_C(E) \rightarrow 0.$$

By assumption,  $S$  does not contain  $-2$ -curves, thus  $C - E$  will be nef. Moreover,  $(C - E)^2 = 2(g - (k+1)) > 0$ , which means that  $C - E$  is big and nef. By Kawamata-Viehweg Vanishing (Theorem 2.15), we have

$$H^1(\mathcal{O}_S(E - C)) \cong H^1(\mathcal{O}_S(C - E))^\vee = 0.$$



This gives the exact sequence of global sections

$$0 \longrightarrow H^0(\mathcal{O}_S(E - C)) \longrightarrow H^0(\mathcal{O}_S(E)) \longrightarrow H^0(\mathcal{O}_C(E)) \longrightarrow 0.$$

$H^0(\mathcal{O}_S(E - C)) = 0$ , since  $C - E$  is effective. Therefore  $h^0(\mathcal{O}_C(E)) = h^0(\mathcal{O}_S(E)) = 2$  by Riemann-Roch. This shows that  $\mathfrak{g}_k^1$  is complete. It remains to show that  $\mathfrak{g}_k^1$  is base point free.

Assume that  $\mathfrak{g}_k^1$  contain a base point  $P$ . Then  $\dim|E|_C - P| = \dim|E|_C|$ , by Lemma 2.10, which means that  $h^0(\mathcal{O}_C(E|_C - P)) = h^0(\mathcal{O}_C(E|_C)) = 2$ . We use the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

where  $\mathcal{O}_P$  is just the skyscraper sheaf at the point  $P$ . Tensor the exact sequence above with  $\mathcal{O}_C(E)$ , so that

$$0 \longrightarrow \mathcal{O}_C(E|_C - P) \longrightarrow \mathcal{O}_C(E|_C) \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

Notice that tensoring with  $\mathcal{O}_C(E|_C)$  does not affect  $\mathcal{O}_P$ , since  $\mathcal{O}_C(E|_C)$  is locally free of rank 1. From the exact sequences above, we obtain the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_C(E)) & & \\ \downarrow \alpha & \searrow \beta & \\ H^0(\mathcal{O}_C(E|_C)) & \xrightarrow{\gamma} & H^0(\mathcal{O}_P). \end{array}$$

$\alpha$  is clearly surjective by Kawamata-Viehweg Vanishing.  $\beta$  is also surjective, because  $H^0(\mathcal{O}_C(E|_C) \otimes \mathcal{I}_P) \subsetneq H^0(\mathcal{O}_C(E))$ . This implies that  $\gamma$  is surjective. On the other hand,  $H^0(\mathcal{O}_C(E|_C - P)) \cong H^0(\mathcal{O}_C(E|_C))$ , which implies that  $\gamma$  cannot be surjective, so we get a contradiction. This shows that  $\mathfrak{g}_k^1$  is base point free. Applying Theorem 6.5, we conclude that  $\text{gon}(C) = k$ .  $\square$



# Chapter 7

## Regular Points in $\text{Hilb}_{d,g,r}$ .

### 7.1 Expected Dimension and the Hilbert Scheme, $\text{Hilb}_{d,g,r}$

Scheme theory was introduced by Alexander Grothendieck in the very famous papers EGA (Éléments de géométrie algébrique) in the period 1960-1967, and are definitely considered as one of the most important contribution to modern mathematics. The idea with scheme theory is to broaden the concept of a variety, which makes scheme theory to an extremely important tool.

Roughly speaking, the Hilbert scheme parametrizes subschemes of a fixed projective space with a prescribed Hilbert polynomial. Our motivation for discussing the Hilbert scheme is that in the next sections we will look at curves on complete intersection K3 surfaces that corresponds to smooth points in the Hilbert scheme. But first some notions and a short introduction.

We will denote the Hilbert scheme of curves  $C \subseteq \mathbb{P}^r$  with  $p_a(C) = g$  and  $\deg(C) = d$  as  $\text{Hilb}_{d,g,r}$ . A component of  $\text{Hilb}_{d,g,r}$  is said to be *regular* if its general point corresponds to a smooth irreducible curve  $C \subseteq \mathbb{P}^r$  such that  $H^1(C, \mathcal{N}_{C/\mathbb{P}^r}) = 0$ . We will show that a regular component of  $\text{Hilb}_{d,g,r}$  is generically smooth of the expected dimension

$$\chi(C, \mathcal{N}_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1), \quad (7.1.1)$$

but first two important results about the lower bound and the upper bound of the dimension of  $\text{Hilb}_{d,g,r}$  at a point. The next two results can be found in [ACG11, p. 33].

**Corollary 7.1.** *The tangent space to  $\text{Hilb}_{d,g,r}$  at a point  $h$  is given by*

$$T_h(\text{Hilb}_{d,g,r}) = H^0(X, \mathcal{N}_{X/\mathbb{P}^r}),$$

where  $X$  is a subscheme of  $\mathbb{P}^r$  of degree  $d$  and genus  $g$ .

Notice that this result shows that  $h^0(X, \mathcal{N}_{X/\mathbb{P}^r})$  is an upper bound for the dimension of  $\text{Hilb}_{d,g,r}$ . The lower bound is given in the following result.

**Proposition 7.2.** *Let  $X$  be a closed local complete intersection subscheme of  $\mathbb{P}^r$ , and let  $h$  be the corresponding point of  $\text{Hilb}_{d,g,r}$ . Then the dimension of every irreducible component of  $\text{Hilb}_{d,g,r}$  at  $h$  is at least*

$$h^0(X, \mathcal{N}_{X/\mathbb{P}^r}) - h^1(X, \mathcal{N}_{X/\mathbb{P}^r}).$$

*Proof.* See [Kol96] or [Ser06]. □

Notice that if  $X$  is a curve, then the lower bound is just the Euler-Poincaré characteristic of the normal bundle on  $X$ . We are now ready to show the expected dimension (7.1.1) of a regular component of  $\text{Hilb}_{d,g,r}$ .

Consider a smooth complete nondegenerate curve  $C \subseteq \mathbb{P}^r$  of degree  $d$  and genus  $g$ . The curve  $C$  corresponds to a point  $[C]$  in  $\text{Hilb}_{d,g,r}$ . By Corollary 7.1 and Proposition 7.2, we have

$$\chi(C, \mathcal{N}_{C/\mathbb{P}^r}) \leq \dim_{[C]} \text{Hilb}_{d,g,r} \leq h^0(C, \mathcal{N}_{C/\mathbb{P}^r}).$$

The Euler-Poincaré characteristic of  $\mathcal{N}_{C/\mathbb{P}^r}$  can be found by using the exact sequence (2.4.4)

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_C \longrightarrow \mathcal{N}_{C/\mathbb{P}^r} \longrightarrow 0.$$

Using that the fact that the tangent sheaf on a variety is equal to the dual of the canonical sheaf and that

$$\deg \mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_C = \deg \mathcal{T}_C + \deg \mathcal{N}_{C/\mathbb{P}^r},$$

$$\deg \mathcal{T}_C = 2 - 2g \text{ and } \deg \mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_C = (r + 1)d$$

gives

$$\deg \mathcal{N}_{C/\mathbb{P}^r} = (r + 1)d + 2g - 2.$$

Now by Riemann-Roch, we have

$$\begin{aligned} \chi(C, \mathcal{N}_{C/\mathbb{P}^r}) &= \deg \mathcal{N}_{C/\mathbb{P}^r} - (r - 1)(g - 1) \\ &= (r + 1)d - (r - 3)(g - 1). \end{aligned}$$

*Remark 7.3.* An interesting observation about  $\chi(C, \mathcal{N}_{C/\mathbb{P}^r})$ , when we look at it as a function of  $g$ , is that for  $r = 2$  it increases with  $g$ ; when  $r = 3$ , it is equal to  $4d$  and is independent of  $g$ , while for  $r \geq 4$  it decreases with  $g$ .

We will end our discussion about Hilbert scheme<sup>1</sup>. In the next sections we will look at cases when  $\dim \text{Hilb}_{d,g,r}$  is exactly  $\chi(C, \mathcal{N}_{C/\mathbb{P}^r})$ .

## 7.2 $K3$ Surfaces With Picard Rank 2

In this section we will study smooth curves of degree  $d$  and genus  $g$  on a  $K3$  surface  $S \subseteq \mathbb{P}^r$ , that corresponds to smooth points in the Hilbert scheme,  $\text{Hilb}_{d,g,r}$ .

The next proposition is a generalization of Proposition 4.1 in [Far01], which says that a curve sitting on a quartic  $K3$  surface in  $\mathbb{P}^3$  corresponds to smooth points in  $\text{Hilb}_{d,g,3}$  if and only if  $d \leq 18$  or  $g < 4d - 31$ , when the  $K3$  surface does not contain  $-2$ -curves. We have generalized this result by just referring to Proposition 4.5 and we have filled in some details in the proof.

**Proposition 7.4.** *Let  $C \subseteq S \subseteq \mathbb{P}^3$  be a smooth curve sitting on a quartic surface such that  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$  with  $H$  being a hyperplane section. Then  $H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) = 0$  if and only if  $d \leq 16$  or  $g < 4d - 32$ .*

*Proof.* We have the following exact sequence

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbb{P}^3} \longrightarrow \mathcal{N}_{S/\mathbb{P}^3} \otimes \mathcal{O}_C \longrightarrow 0, \quad (7.2.1)$$

where  $\mathcal{N}_{S/\mathbb{P}^3} \otimes \mathcal{O}_C \cong \mathcal{O}_C(4)$  and  $\mathcal{N}_{C/S} \cong K_C$ . We claim that there is an isomorphism  $H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) \cong H^1(C, \mathcal{O}_C(4))$ . Suppose the contrary. Since  $H^0(C, \mathcal{O}_C(4)) = 0$ , we get the injective map  $H^1(C, K_C) \rightarrow H^1(C, \mathcal{N}_{C/\mathbb{P}^3})$ . We will show that this provides a section  $\sigma \in \mathcal{H}om(\mathcal{N}_{C/\mathbb{P}^3}, K_C)$ . Using the isomorphisms

$$H^1(C, K_C) \cong H^0(C, \mathcal{O}_C)^\vee \text{ and } H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) \cong H^0(C, \mathcal{N}_{C/\mathbb{P}^3}^\vee \otimes K_C)^\vee.$$

The dual of the sequence

$$0 \longrightarrow H^0(C, \mathcal{O}_C)^\vee \longrightarrow H^0(C, \mathcal{N}_{C/\mathbb{P}^3}^\vee \otimes K_C)^\vee \longrightarrow H^1(C, \mathcal{O}_C(4)) \longrightarrow 0,$$

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<sup>1</sup>For more information about the Hilbert scheme, see Chapter IX in [Ser06] and/or Chapter VI.2.2 in [EH01].

gives

$$0 \longrightarrow H^1(C, \mathcal{O}_C(4)) \longrightarrow \mathcal{H}om(\mathcal{N}_{C/\mathbb{P}^3}, K_C) \longrightarrow \mathbb{C}.$$

This gives a section  $\sigma \in \mathcal{H}om(\mathcal{N}_{C/\mathbb{P}^3}, K_C)$ . On the other hand, if we take the dual of the exact sequence (7.2.1), we get

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om(\mathcal{O}_C(4), \mathcal{O}_C) &\longrightarrow \mathcal{H}om(\mathcal{N}_{C/\mathbb{P}^3}, \mathcal{O}_C) \longrightarrow \mathcal{H}om(K_C, \mathcal{O}_C) \\ &\longrightarrow \mathcal{E}xt^1(\mathcal{O}_C(4), \mathcal{O}_C) \longrightarrow \dots \end{aligned}$$

We now that  $\mathcal{E}xt^1(\mathcal{O}_C(4), \mathcal{O}_C) = 0$ , since  $\mathcal{O}_C(4)$  is locally free by [Har77, Chapter III, Exercise 6.5 (a)]. We have a section

$$\sigma' \in \mathcal{H}om(K_C, \mathcal{N}_{C/\mathbb{P}^3}).$$

The composition of  $\sigma$  and  $\sigma'$  gives the splitting of the dual sequence above. Dualizing again provides a splitting of the bundle sequence (7.2.1). Now Theorem 2.16 implies that  $C$  is a complete intersection with  $S$ . This is a contradiction. Therefore we have  $H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) \cong H^1(C, \mathcal{O}_C(4))$ . Using Proposition 4.5, we find that  $H^1(C, \mathcal{O}_C(4))$  is nonspecial if and only if  $d \leq 16$  or  $g < 4d - 32$ .  $\square$

In general, it can be shown that if  $S$  is a complete intersection K3 surface of one of the three types (4), (2, 3) or (2, 2, 2) with Picard group of rank 2 and where  $H|_C$  is nonspecial, then  $\text{Hilb}_{d,g,r}$  is smooth at all points representing curves  $C' \in |C|$ . In the next lemma we show under what circumstances smooth curves on the three types of complete intersection K3s corresponds to smooth points in the Hilbert scheme.

**Lemma 7.5.** *Let  $S$  be a complete intersection K3 surface, with  $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1) \oplus \mathbb{Z}C$ , in  $\mathbb{P}^r$  and  $C_0$  a smooth genus  $g$  curve of degree  $d$  on  $S$ . Then if*

$$i) \ d \leq 4(r-1) \text{ and } 2d > 4(r-1) + g;$$

$$ii) \ d \leq 8(r-1) \text{ and } 4d > 16(r-1) + g, \text{ then}$$

$\text{Hilb}_{d,g,r}$  is smooth at all points representing curves  $C \in |\mathcal{O}_S(C_0)|$  where  $S$  is either  $S_{2,3}$  or  $S_{2,2,2}$  in i) and  $S_4$  in ii).

*Proof.* Let  $S$  be a K3 surface which is a complete intersection of type  $(a_1, a_2, \dots, a_{r-2})$  in  $\mathbb{P}^r$  and that  $\mathcal{O}_{C_0}(\min\{a_i\})$  is nonspecial. Then  $\mathcal{O}_C(\min\{a_i\})$

for  $C \in |\mathcal{O}_S(C_0)|$ , by Lemma 4.11. We know that when  $S$  is a complete intersection the normal bundle  $\mathcal{N}_{S/\mathbb{P}^r}$  splits [Har77, Chapter II, Exercise 8.4], so  $\mathcal{N}_{S/\mathbb{P}^r} \cong \bigoplus_{i=1}^{r-2} \mathcal{O}_S(a_i)$ . Restricting the normal bundle on  $S$  to the smooth curve  $C$  gives

$$\begin{aligned} \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C &\cong \left[ \bigoplus_{i=1}^{r-2} \mathcal{O}_S(a_i) \right] \otimes \mathcal{O}_C \\ &\cong \bigoplus_{i=1}^{r-2} \mathcal{O}_S(a_i) \otimes \mathcal{O}_C \\ &\cong \bigoplus_{i=1}^{r-2} \mathcal{O}_C(a_i). \end{aligned}$$

Direct sum commutes with cohomology, so

$$H^1(C, \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C) \cong \bigoplus_{i=1}^{r-2} H^1(C, \mathcal{O}_C(a_i)).$$

Since  $\mathcal{O}_C(\min\{a_i\})$  is nonspecial, we have that  $H^1(C, \mathcal{O}_C(\min\{a_i\})) = 0$ . Hence

$$H^1(C, \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C) = 0. \quad (7.2.2)$$

The exact sequence (7.2.1) in  $\mathbb{P}^r$  becomes,

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbb{P}^r} \longrightarrow \bigoplus_{i=1}^{r-2} \mathcal{O}_C(a_i) \longrightarrow 0.$$

Using (7.2.2) and the long exact sequence of cohomology, we get

$$\dots \longrightarrow \bigoplus H^0(C, \mathcal{O}_C(a_i)) \longrightarrow H^1(C, \mathcal{N}_{C/S}) \longrightarrow H^1(C, \mathcal{N}_{C/\mathbb{P}^r}) \longrightarrow 0.$$

Since the composition of

$$H^0(S, \mathcal{N}_{S/\mathbb{P}^r}) \longrightarrow H^0(C, \mathcal{N}_{S/\mathbb{P}^r} \otimes \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{N}_{C/S})$$

is surjective by Theorem 4.14, we get  $H^1(C, \mathcal{N}_{C/\mathbb{P}^r}) = 0$ . The necessary numerical conditions for  $\mathcal{O}_{C_0}(2)$  and  $\mathcal{O}_{C_0}(4)$  to be nonspecial are given in Proposition 4.5.  $\square$

### 7.3 $K3$ Surfaces With Picard Rank 3

The first result is the analogue of Proposition 4.5 when  $S$  is equipped with a Picard group of rank 3, with some restrictions on the triplet  $(d, g, r)$ . The proof is the same as the first part proof of [Knu02, Proposition 1.3]. Because of our restrictions on  $(d, g, r)$ ,  $S$  contains no rational curves and the problem becomes very easy.

**Lemma 7.6.** *Let  $l \geq 1$  be an integer,  $d$  is divisible by 4 and  $g, r$  odd. We can find  $S \subseteq \mathbb{P}^r$  with a Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , which contains no  $-2$ -curves, and  $C$  such that  $h^1(C', \mathcal{O}_{C'}(l)) = 0$  for all  $C' \in |C|$  if and only if*

$$d \leq 2(r-1)l \text{ or } dl > nl^2 + g.$$

*Proof.* The existence of  $S$  with  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$  is given by Proposition 6.1 and the embedding into  $\mathbb{P}^r$  is given by Proposition 6.3, hence the numerical conditions on the triplet  $(d, g, r)$ . Let  $C' \in |C|$ . By the exact sequence

$$0 \longrightarrow \mathcal{O}_S(lH - C) \longrightarrow \mathcal{O}_S(lH) \longrightarrow \mathcal{O}_{C'}(lH) \longrightarrow 0,$$

and since  $H$  is ample, we get that

$$H^1(S, \mathcal{O}(lH)) = H^2(S, \mathcal{O}(lH)) = 0,$$

by using Kodaira vanishing theorem (Theorem 2.14). Furthermore, using Serre duality

$$H^1(C', \mathcal{O}_{C'}(lH)) \cong H^2(S, \mathcal{O}_S(lH - C)) \cong H^0(S, \mathcal{O}_S(C - lH))^\vee,$$

hence  $h^1(S, \mathcal{O}_{C'}(lH)) = h^0(S, \mathcal{O}_S(C - lH))$ .

If  $d \leq 2nl$ , the  $(C - lH) \cdot H = d - 2nl \leq 0$ . This implies that  $h^0(S, \mathcal{O}_S(C - lH)) = 0$ , since  $H$  is ample.

If  $d > 2nl$  and  $dl \leq nl^2 + g$ , we have  $(C - lH)^2 \leq -2$  and  $(C - lH) \cdot H > 0$ . By Riemann-Roch (3.1.2),  $C - lH > 0$ . Let  $d > 2nl$  and  $dl > nl^2 + g$  and assume that there is an element  $D \in |C - lH|$ . Then  $D^2 < -2$ , so  $D$  has to contain an irreducible curve  $\Gamma$  such that  $D \cdot \Gamma < 0$  and  $\Gamma^2 = -2$ . But since  $S$  does not contain rational curves, we are done.  $\square$

*Remark 7.7.* It is important to notice that this theorem only holds for curves with degree divisible by 4 and odd genus lying on  $K3$  surfaces in a projective space with odd dimension. A consequence of this, is that we cannot find the numerical conditions for when a line bundle on  $S_{2,3}$  is nonspecial.



In the next corollary we find the numerical conditions for when a smooth curve on the complete intersections  $S_4 \subseteq \mathbb{P}^3$  and  $S_{2,2,2} \subseteq \mathbb{P}^5$  with  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$  corresponds to a smooth point in the Hilbert scheme. The proof is the same as in Lemma 7.5.

**Corollary 7.8.** *Let  $S_4 \subseteq \mathbb{P}^3$  and  $S_{2,2,2} \subseteq \mathbb{P}^5$  be the standard complete intersections, both equipped with the Picard group  $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ . Moreover, assume that  $S$  contains no  $-2$ -curves. If*

- i)  $C \subseteq S_4$  with  $d$  divisible by 4,  $g$  odd and  $d \leq 16$  or  $4d > 32 + g$ , then  $[C] \in \text{Hilb}_{d,g,3}$  is a regular point.
- ii)  $C \subseteq S_{2,2,2}$  with  $d$  divisible by 4,  $g$  odd and  $d \leq 8$  or  $2d > 8 + g$ , then  $[C] \in \text{Hilb}_{d,g,5}$  is a regular point.

*Proof.* Let  $S$  be the quartic hypersurface in  $\mathbb{P}^3$ . The normal bundle splits, so

$$\mathcal{N}_{S/\mathbb{P}^3} \cong \mathcal{O}_S(4). \quad (7.3.1)$$

Using the same standard exact sequence (7.2.1) as in the rank 2 case and the isomorphism (7.3.1) one has

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbb{P}^3} \longrightarrow \mathcal{O}_C(4) \longrightarrow 0.$$

If  $d \leq 16$  or  $4d > 32 + g$  with  $d$  even and  $g$  odd, then the line bundle  $\mathcal{O}_C(4)$  is nonspecial, by Corollary 7.6. The long exact sequence gives

$$\dots \longrightarrow H^0(C, \mathcal{O}_C(4)) \longrightarrow H^1(C, \mathcal{N}_{C/S}) \longrightarrow H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) \longrightarrow 0.$$

Using Lemma 4.11, we get that  $H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) = 0$ . For the complete intersection of three quadric hypersurfaces in  $\mathbb{P}^5$ , we use exactly the same procedure as in the previous case. Now  $\mathcal{N}_{S/\mathbb{P}^5} \cong \mathcal{O}_S(2)^{\oplus 3}$ . If  $d \leq 8$  or  $2d > 16 + g$  with  $d$  even and  $g$  odd, then  $H^1(C, \mathcal{O}_C(2)) = 0$ , again by Corollary 7.6. Hence

$$\dots \longrightarrow H^0(C, \mathcal{O}_C(2))^{\oplus 3} \longrightarrow H^1(C, \mathcal{N}_{C/S}) \longrightarrow H^1(C, \mathcal{N}_{C/\mathbb{P}^5}) \longrightarrow 0.$$

We get  $H^1(C, \mathcal{N}_{C/\mathbb{P}^5}) = 0$ , again by Lemma 4.11.  $\square$



# Chapter 8

## Further Work

Finally, we will suggest some ideas and other possible approaches to new results and generalizations of what has been done in this thesis.

### 8.1 Unfinished/Unsolved Problems

In this section we will look at some problems I have been working on and not been able to finish in time.

Exceptional curves on  $K3$  surfaces of Picard rank 2: If  $C$  is a smooth curve of degree  $d$  and genus  $g$  satisfying Theorem 4.3, we have shown, in Corollary 4.9, that exceptional curves occurs only in situation iv). I tried to find all exceptional curves, but the problem was to show the nonexistence of the line bundle  $B$  in the generalized ELMS examples (Example 3.13).

Theorem 5.1: This theorem would have been optimal if we could get rid of  $d \geq r^2 + r$ , i.e. assume  $d > 0$  instead. If we had done this, we must used Corollary 4.9 which eliminates the generalized ELMS examples and the Donagi-Morrison example. In the proof, I show that the assumption  $d \geq r^2 + r$  together with the Brill-Noether number  $\rho(d, g, r) = g - (r+1)(g-d+r) < 0$  implies  $d \leq g - 1$ . I have just used that  $d \leq g - 1$  only once in the proof (see p. 56). The consequence of assuming  $d > 0$ , is that we would have had more candidates for the gonality, i.e. the endpoints (5.3.11) in addition to  $N_r$ , which would also lead to a lot more calculations in the proof of Corollary 5.3.

The very ampleness of  $H$  in the rank 3 case: The first problem I had to take care of, was the existence of a very ample hyperplane section. Using

Lemma 4.4 and that  $D \in \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , we get  $D^2 = 0, -2$  is a ternary quadratic equation and  $D.H = 0, 1, 2$  is a linear equation in three variables. According to Lemma 4.4 we must eliminate the cases where  $D^2 = 0$ ,  $D.H = 1, 2$  and  $D^2 = -2$ ,  $D.H = 0$ . This is a hard problem numerically, so we made some restrictions on the triplet  $(d, g, r)$ . Another approach, was to search for articles about solutions of ternary quadratic forms, i.e. find candidates (other than the restrictions already made) of  $d$ ,  $g$  and  $r$  such that  $\frac{D^2}{2} \neq 0, -1$ .

The nefness of  $C$  and  $E$  in the rank 3 case: The second problem concerning the rank 3 case, was to show that both  $C$  and  $E$  are nef. Again I tried to solve this numerically. One approach, which I tried, is assuming that there exists irreducible rational curves  $\Gamma$  and  $\Gamma'$  on  $S$  such that  $C.\Gamma < 0$  and  $E.\Gamma' < 0$ , and derive a contradiction. I spend a lot of time working on this, and I made a lot of progress in the end, but because the time limit we had to make priorities. I used the Picard-Lefschetz reflections, described on page 67, on  $C$  and  $E$ . That is,  $C \mapsto C'$  and  $E \mapsto E'$ , and from there try to show that  $|\text{disc}(H, C', E)| < |\text{disc}(H, C, E)|$ , and the same with  $E$ . If one can do this, we are done, because  $H, C$  and  $E$  generates the Picard group, thus  $\text{disc}(H, C, E)$  divides  $\text{disc}(H, C', E)$  which gives a contradiction.

The complete intersection  $S_{2,3}$  in  $\mathbb{P}^4$ : Since I had to make restrictions on  $d$ ,  $g$  and  $r$ , we couldn't embed this surface into  $\mathbb{P}^4$  and therefore not study the curves on this surface.

## 8.2 Some Ideas

Deform the Kummer surface: Recall Example 3.25, where we looked at the Kummer surface  $S_0 = \text{Km}(C_1 \times C_2)$ , where  $C_1$  and  $C_2$  are smooth elliptic curves. The idea is to find three divisors  $H$ ,  $C$  and  $E$  on  $S_0$  where all three are nef and base point free with the intersection properties:  $H^2 = 2r - 2$ ,  $C^2 = 2g - 2$ ,  $E^2 = 0$ ,  $H.C = d$ ,  $H.E = \alpha$  and  $E.C = k$ . Now we could deform  $S_0$  with Picard rank 18 to a  $K3$  surface with  $S_t$  with Picard group  $\text{Pic}(S_t) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ , where base point freeness and nefness is preserved. If the procedure above is done correctly, the next step is to show that  $\text{gon}(C)$  is equal to  $k$ .

Other parities of  $(d, g)$ : This idea comes the proof of [Far01, Theorem 1], where the parity problem in the rank 2 case, is taken care of. Theorem 6.3 holds when  $d$  is divisible by 4 and  $r, g$  are odd. To make the idea clear,

we look at the quartic  $S_4$  in  $\mathbb{P}^3$ . By Theorem 6.3 and Corollary 7.8 there exists a nondegenerate curve  $C \subseteq \mathbb{P}^3$  of degree  $d$  divisible by 4 and odd genus  $g > (k-1)(k-2)$ , with  $\text{gon}(C) = k$  and  $H^1(C, \mathcal{N}_{C/\mathbb{P}^3}) = 0$ , where  $C$  is on the quartic  $S_4$  and  $\text{Pic}(S_4) = \mathbb{Z}H \oplus \mathbb{Z}C \oplus \mathbb{Z}E$ . If we attach a 2 or 3-secant line or a 4-secant conic, we may be able to show that it works for other parities of  $(d, g)$  as well.

Rational curves in the rank 3 case: If we can show that  $H$  is very ample,  $C$  and  $E$  are nef we can create powerful generalizations of all results concerning  $K3$  surfaces of rank 3. Maybe we have to come up with other ideas, but I believe that one can solve this problem numerically and perhaps with the help of computer, just to avoid a lot of time consuming calculations.



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